LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER

ACTUARIAL SCIENCE

MASTER´S FINAL WORK

DISSERTATION

INDIVIDUAL AND NATIONAL FINANCIAL DECISION-MAKING ON RETIREMENT : ILLUSTRATION OF REPUBLIC OF KOREA

KANG WANSUB

OCTOBER-2020

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SUPERVISION: AGNIESZKA IZABELLA BERGEL

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Abstract

Not to mention of all times and countries, life span is the main subject in society. Reaching at age 60 was regarded as a great event and people specially celebrated. In the twenty-first century, it is common to reach age 65. The paradigm of life cycle is gravely changed and we face somewhat a different world beyond imagination.

Throughout the thesis, various financial decision-making models on retirement are analyzed and applied to the statistics of Republic of KOREA. The basic principles are the time value of money and the probability encompassing PV (Present Value) and APV (Actuarial Present Value).

When it comes to individual dimension of financial decision-making, we present Leonardo Fibonacci and Edmond Henry annuity equation as historical cases. Then, exchange ratio and ruin probability reckon remarkable numbers and draw practical implications.

As for national dimension of financial decision-making, above all mortality rate and life expectancy at age 65 are projected with reflecting mortality improvement. In order to assess public pension, projection on population and dependency ratio are preceded beforehand. With the result of published reports, actuarial valuation and sustainability of public pension are discussed.

Besides, the invention of "Polynomial Power Series Triangle" (Eulerian number) is illustrated with the application of geometric distribution.

The thesis intends to help individuals to make an informed decision. For government, it suggests taking full-fledging measurements for existing pension system of Republic of Korea in 2023 to take the impact of swift demographic change in decades into consideration.

Keywords: Individual financial decision-making; National financial decision-making; Public pension in Republic of KOREA; Polynomial power series triangle (Eulerian number)

Resumo

O ponto fulcral da sociedade, independentemente da época ou país, é o tempo de vida. Antigamente, atingir os 60 anos era motivo de celebração especial. Agora, no século XXI, é comum atingir os 65 anos. O paradigma do ciclo de vida tem mudado bruscamente e encontramos numa situação em tempos inimaginável.

Durante a tese, analisam-se vários modelos de decisão financeira da reforma da República da Coreia. Os princípios básicos da análise são o valor temporal do dinheiro e as probabilidades associadas ao valor atual e ao valor atual atuarial.

No que diz respeito à decisão financeira relativo ao indivíduo apresentamos a equação de anuidade de Leonardo Fibonacci e Edmond Henry como casos históricos. Posteriormente, aplica-se a taxa de troca e a probabilidade de ruína para chegar a conclusões uteís.

Relativamente à decisão financeira numa dimensão nacional, a taxa de mortalidade e a esperança média de vida aos 65 anos são projetadas considerando melhorias na mortalidade. Para avaliar a sustentabilidade de pensões públicas, projeta-se a população e a respetiva taxa de dependência. Por fim discutimos a sustentabilidade das pensões públicas é discutida tendo em consideração reportes publicados e avaliações atuariais.

É ainda demonstrado o triângulo de séries de potências com polinómios (Número de Euler) com a aplicação da distribuição geométrica.

Com esta tese queremos ajudar indivíduos a tomar decisões informadas. Ao governo, sugere medidas a tomar que permitam ao sistema de pensões adaptar-se às mudanças demográficas rápidas que se esperam para as próximas décadas.

Palavras-chave: Tomada de decisão financeira individual; Tomada de decisão financeira nacional; Pensão pública na República da Coreia; Triângulo de potências polinomial (Número de Euler)

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As if by a slip of fate, I came to Portugal for master's degree of actuarial science in 2017 and bid farewell to ISEG (Instituto Superior de Economia e Gestão) in 2020.

Come to think of it, the most difficult task was master's thesis. It seemed never-ending story, but I finally see the light of day.

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It has been a challenging journey of life from studying to working abroad. I sincerely appreciate my wife for inspiring and supporting me. In addition, I warmly show gratitude toward my family.

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1. Introduction

Beyond imagination, the world in 2020, after 20 years of millennium, has gone through unprecedented change compared to one century ago. Taking into account the circumstances of World Wars, life expectancy in 1900 amounted to 31 years and in 1950 it was 48 years (WHO, 2010). In 2017, world average life expectancy at birth was recorded on the level of 72.4 years (WB, 2018), which is the highest level in mankind history and it keeps growing. In particular, the average life expectancy at the age of 65 shows interesting developments. In Korea, statistically a man who reached age 65 lives for 18.6 years more, whereas for a woman it is 22.7 years on average (KOSIS, 2017a). Supposing that retirement age is 65, there is a considerable time remained to the pensioners. The unparalleled change of the world leaves us the matter how to deal with so-called “longevity risk” in both individual and national basis.

In 1883 Otto Von Bismarck introduced pension system and laid the ground to establish social security. At that time, nobody could imagine that pension system will face unsustainable development at risk.

Around the new millennium, so-called “New Normal” era featuring low growth, low interest rate, low birth rate, and low employment rate is normal in advanced countries and is prevalent in all around the world. Urbanization accelerated demographic transition from big family to nuclear family, decomposing traditional structure of family what consequently led to low birth rate. Education also indirectly had an impact on low birth rate. These multi-facet factors are intertwined with each other, thus causing dramatic change of demography and therefore the sustainability of pension system is under question.

Along with the trends, pension system becomes a key agenda for governments as demographic transition is accelerated. From 1990s, many advanced countries began to deal with pension system in depth. Each government has taken measures such as gradually postponing retirement age, retrenching pension benefits or raising contribution rates. Even structural reform has been implemented in several counties. In Korea, target replacement rate of public scheme decreased from 49.5% to 40% between 2009 and 2028 [OECD (2013), p36]. In addition, a new basic pension benefit was introduced in July 2014 [OECD (2015), p41]. The core issue is that the contribution rate has been fixed to 9% of income since 2003 when the first pension valuation report was published.

Table 1: Demographic old-age dependency ratios: Historical and projected values, 1950-2075 (Selected countries)

	1950	*1975*	*2000*	*2015*	*2025*	*2050*	*2075*
Korea	6.3	8.2	11.2	19.4	31.7	72.4	78.8
China	8.5	8.8	11.4	14.5	22.3	47.9	58.8
France	19.5	24.5	27.3	33.3	40.9	52.3	55.8
Germany	16.2	26.5	26.5	34.8	41.4	59.2	63.1
Poland	9.4	17.1	20.1	24.3	36.4	60.8	73.3
Portugal	13.0	19.6	26.8	34.6	42.4	73.2	77.6
Switzerland	15.8	21.5	24.9	29.0	35.4	54.6	58.1
United Kingdom	17.9	25.5	27.0	31.0	35.9	48.0	53.0
United States	14.2	19.7	20.9	24.6	32.9	40.3	49.3

Source: United Nations (UN), World Population Prospects – 2017 Revision

The above table shows old-age dependency ratios from 1950 to 2075 both historical and projected values from various countries. The demographic old-age dependency ratio is defined as the number of individuals aged 65 and above per 100 people of working age which is between 20 and 64 years old [OECD (2017), p125]. Korea has experienced drastic demographic transition. In 1950s Korea was one of the youngest societies in the world, however in 50 years Korea is expected to be one of the oldest societies in the world. The reason for this is the combination of severe ageing transition as well as low fertility rate. Life expectancy in Korea for man and woman is 79.5 and 85.6 respectively and overall it amounts to 82.7 which put Korea on 9th highest rank among all countries (WB, 2018). Moreover, Korea's fertility rate significantly dropped from 4.50 in 1970 to 1.2 in 2016 [SOA (2018), p10]. Korea recorded the lowest Total Fertility Rate (TFR) 0.977 in 2018 which is far below replacement rate of 2.1 required to maintain the stable population (KOSIS, 2018). Hence, the shape of population in Korea will be quickly transformed into an inverted triangle within coming decades.

Note that Korean pension system came into existence in 1987. After 30 years the accumulated National Pension Fund (NPF) reserve reached 621.7 trillion Korean Won, which is third largest fund in the world [NPRI (2017), p22]. However, the reserve will be depleted by 2060 after reaching peak in 2043 according to actuarial valuation [NPS (2013), p20]. In particular, the poverty of aged 66 and above equaled to 45.7% in Korea, being the highest percentage among OECD (Organization for Economic Cooperation and Development) countries in 2017 [OECD (2017), p137]. Due to this difficult situation of Korea, young generation will bear heavy burden for supporting old generation in the future. At the same time, old generation above 66 years old will also experience hardships.

As longevity risk is not solely solved by decision made on national level, it becomes the seminal issue in individual life more and more. Each individual faces serious challenge to prepare for retirement plan by oneself. That is, each individual confronts making a financial decision such as saving, spending, and investment depending on personal circumstances such as age, health, and mortality. Reasonable decision on the level of savings and consumptions is required to sustain a decent standard of living. However, SOA (Society Of Actuaries) survey presents that only 12 percent of respondents have started retirement planning [SOA (2018), p26]. Moreover, 77 percent of them do not have formal written plans on how to allocate their portfolio of income, assets, and expenses during retirement [SOA (2018), p29]. There is a huge gap between the change of reality and both preparation and recognition of retirement.

1.1 Motivation

Equipped with better recognition of time value of money and probability, individual will make a wiser financial choice at each phase and even design smooth expenditure plan along with life cycle. The policymaker finds the best solution based on not only good quality of statistics, but also swift and precise model. Each part can create constructive incentives to shape virtuous cycle. Therefore, the paper intends to provide suggestive points for individual to make a reasonable decision on current savings and consumption level in a long run perspective and policymaker to design insightful policy.

1.2 Organization of the thesis

In Chapter 2, practical models are presented belonging to individual-level such as Leonardo Fibonacci equation, Edmond Henry annuity equation, exchange ratio, and ruin probability. In Chapter 3, public pension, which is the symbol of welfare country, is analyzed in depth, discussing issues on national dimension such as life expectancy and demographic transition. In Chapter 4, results and interpretations on financial decision-making of micro and macro level are summarized. Last not but least, the beauty of mathematics remains elegant in Appendix.

2. Individual Financial Decision-making on Retirement

2.1 Introduction-illustration of Republic of KOREA

On individual level, the primary question is how much money does one need to save until retiring at age 65 to maintain dignified standard of living for rest of one's life? Certainly, more money leads to a better standard of living in general. However, income is limited and thus limited budget should be allocated between consumption at present and saving for future. A rational person in economics tries to find the optimal allocation of resources. In this section, we investigate several models which give more insight into individual decision-making situation.

2.2 Leonardo Fibonacci Equation

We first present the oldest mathematical model inspired by Leonardo Fibonacci (1170~1250). With the adoption of log function, it is still used even now as a basic reference to calculate required retirement fund for dignified standard of living [See Milevsky (2012), p7].

$$t = \frac{1}{r} ln\left(\frac{c}{c - Wr}\right) \tag{2.2.1}$$

where *t* is life expectancy of the fund, *r* is real interest rate, *c* is the amount of expenditure per year, and *W* is expected retirement fund. Real interest rate is the lending interest rate adjusted for inflation as measured by the GDP (Gross Domestic Product) deflator (WB, 2017). Underlying assumptions are as follows:

- there will be no further contributions;
- the retirement fund *W* will accumulate interest at an annual fixed rate of *r*;
- a fixed annual amount *c* will be withdrawn from the retirement fund *W* each year.

Proof. See Appendix A1.

In order to find the pattern of how life expectancy *t* and real interest rate *r* affect expected retirement fund *W*, the following formula is derived (See Appendix A1).

$$W = \frac{1}{r}(c - \frac{c}{e^{tr}}) \tag{2.2.2}$$

Technical Analysis. See Appendix A2.

The negative correlation between real interest rate *r* and expected retirement fund *W*, the positive correlation between life expectancy *t* and expected retirement fund *W* is confirmed by numerical illustration in Table 2.

Table 2: Expected retirement fund ratio depending on historical real interest rate and woman life expectancy at age 65 (*c*=1)

		1997	***2002***	***2007***	***2012***	***2017***
	i/e₆₅	***17.9***	***18.5***	***20.2***	***21.5***	***22.7***
1997	***7.5%***	9.85	10.00	10.40	10.67	10.90
2002	***3.6%***	13.20	13.51	14.36	14.97	15.51
2007	***4.1%***	12.72	13.01	13.79	14.35	14.83
2012	***4.3%***	12.48	12.75	13.49	14.02	14.48
2017	***1.2%***	16.13	16.61	17.96	18.98	19.90

Retirement fund ratio is the ratio between the expected retirement fund *W* and the amount of expenditure per year *c*. From 1997 to 2017, the real interest rate shown in Table 2 has downward trend as economic growth slows down. At the same time, current generation has on average longer life than previous generation as presented in the Table 2. In result, the combination of two factors requires 19.90 expected retirement fund ratio in 2017 which is more than double compared to 9.85 in 1997.

In other words, let us take into consideration those two mentioned circumstances in 1997 and 2017, in both cases annual payment is equal to 1 as unit amount. In 2017 the women at age 65 is expected to live for 22.7 years and the interest rate is equal to 1.2%, therefore accumulated requirement fund should be 19.9. However, in 1997 the woman's life expectancy at age 65 is 17.9 and real interest rate is 7.5%, the required retirement fund was only 9.85.

According to KReIS (Korean Retirement and Income Study) panel survey in 2017, the suitable amount of retirement expenditure is 2.37 million for marriage couple and 1.45 million Korean Won for single person per month. When converting to yearly basis, the amount equals to 28.44 and 17.4 million Korean Won respectively. Based on WB (WorldBank) data, real interest rate of Korea amounts to 1.2% in 2017. It needs to be considered that life expectancy at age 65 for man and women is 18.6 and 22.7 respectively (KOSIS, 2017a). Considering the fact that married couple of the same age is the most common case (15.4%), such assumption is adopted for calculation of joint life expectation.

Applying the survival rate of man and woman, curtate expectation for joint life is 15.4, meaning that both man and woman are expected to live for 15.4 years. Using these values in the formula, the required amount of post retirement fund for decent standard of life is 290.5, 346.3, and 400.3 million Korean Won for single man, single woman, and married couple respectively. In USD terms, it is roughly 264, 331, 364 thousand dollars (KRW/USD exchange rate is 1100). As the amounts seem overwhelming in general terms, it is reasonable to make sure that these numbers are without any kind of external cash inflows. Taking into consideration that the target rate of disposable income at age 65 is commonly 70% in welfare society, 30% of the required post retirement fund should be prepared. In the ideal scenario, the required amount for individual would change to 87.15, 103.89 and 120.09 million Korean Won for each case mentioned above.

When expected retirement fund *W* is proportional to the amount of expenditure per year *c*, that is *W*=*ck*, *k* is scalar, the formula becomes:

$$t = \frac{1}{r} ln\left(\frac{c}{c - ckr}\right) = \frac{1}{r} ln\left(\frac{1}{1 - kr}\right), \qquad k = \frac{1}{r}(1 - \frac{1}{e^{tr}}) \tag{2.2.3}$$

The retirement fund is actually k times of yearly unit expenditure. Therefore, k is retirement fund ratio. The coefficient k is not constant with time. In a year, the value of k decreases from (16.69, 19.90, 14.08) to (15.89, 19.13, 13.24) for single man, woman and couples at age 66.

The model assumes that real interest rate and amount of expenditure per year are constant for whole period set at starting point time 0. If there are multiple interest rates or amount of expenditure per year for whole period, we can split the period and calculate each wealth with a discount factor until starting point time 0.

Let us look at it from another perspective. The level of retirement fund is expectation value because one of the inputs is life expectancy at age 65. The calculation of life expectancy is as follows. The survival probability that a life age *x* will survive to age *x+1* is matched with the ratio of the number of survivor at age *x+1* and *x* from life table data. That is, survive rate is derived from census, i.e. whole population, where the number of real survivors and deaths is reflected. Hence, survival rate is kind of expectation value featuring the principle of average value. Once the probability of life table is established, we can basically construct the probability table that a life age *x* will survive to age *x+t* until *w*. Finally, life expectancy is the average of future lifetime weighted by probability of each future lifetime from *1* to *w-x* years.

Let us describe the probability of ruin in life. We define that a person is ruined when the wealth of the person is completely depleted. If Korean man at the age of 65 secures 16.69 times of yearly expenditure, what is the probability of ruin in his life? Ruin happens when the person outlives the life expectancy at age 65. That is, the probability of ruin is equal to the probability that the person will survive to age 83.6 (65+18.6=83.6). According to statistics in Korea (KOSIS, 2017a), the probability that the person will survive to age 83 is 0.50582 and age 84 is

0.45866. If we choose linear interpolation method connected to Uniform Distribution of Deaths (UDD), the solution is $p_{65,18.6} = p_{65,18} * 0.4 + p_{65,19} * 0.6 = 0.47752$.

$$\text{Uniform Distribution of Deaths (UDD): } p_{x,t} = p_x * (1-t) + p_{x+1} * t, t \in (0,1) \quad (2.2.4)$$

where x is age, p_x is survival rate at age x for one year, $p_{x,t}$ is conditional probability of an x-year-old surviving t more years. Note that $p_{x,t}$ is an equivalent to actuarial notation ${}_tp_x$.

On the other hand, if you have less retirement fund than threshold amount above, it does not mean you will be ruined with certainty but it means that the probability of ruin is increased. In probability world, dramatic case is possible where a person who has larger amount of retirement fund than threshold turned out to be ruined and vice versa, depending on the realized life expectancy.

2.3 Edmond Henry Annuity Equation

We broach the question that if you can choose the option between life-long annuity and lump sum, what should be the criterion of the decision? In spite of various factors, APV (Actuarial Present Value) principle is the main criterion on financial decision-making. To calculate the value of annuity, the following actuarial notation is used as below. Annuity factor depends on mortality and interest rate. Edmond Henry (1656~1742) introduced annuity factor based on the mortality table of Breslau, currently Wrocław, Poland [Heywood (1985), p282].

$$a_x = v^1 p_x + \cdots + v^{w-x} p_{x,w-x} = \sum_{i=0}^{w-x-1} \prod_{j=0}^{j=i} v^{i+1} p_{x+j} = \sum_{t=1}^{w-x} v^t p_{x,t}, \quad v = \frac{1}{1+r} \quad (2.3.1)$$

where x is age, r is real interest rate, v is the associate discount factor, w is the limiting age, p_x is survival rate at age x for one year, $p_{x,t} = \prod_{j=0}^{t-1} p_{x+j}$ is conditional probability of an x-year-old surviving t more years. Note that $p_{x,1} = p_x$.

Coming back to the definition of curtate expectation for single life, i.e. expectation for time until death, annuity is the summation of expectation for time until death weighted by time. Depending on real interest rate and age, the numerical illustration of annuity is presented below. As for mortality rate, Korean statistics (KOSIS, 2017a) is used in all the calculations. When interest goes up, the value of annuity goes down because the higher real interest rate r leads lower discount factor v. As a person gets older, the annuity factor is decreased in slower pace. Furthermore, it is evident that woman has higher annuity factor than man due to higher survival rate.

Table 3: Annuity factor at age 65 depends on real interest rate and survival rate

	Interest 1%	***Interest 1.25%***	***Interest 1.5%***	***Interest 1.75%***	***Interest 2%***
Man	17.22	16.80	16.39	16.00	15.62
Woman	20.57	19.99	19.43	18.90	18.39

Table 4: Annuity factor at real interest rate 2% depends on age and survival rate

	Age 65	***Age 66***	***Age 67***	***Age 68***	***Age 69***
Man	15.62	15.07	14.53	13.98	13.43
Woman	18.39	17.81	17.22	16.62	16.03

Annuity factors in the two tables are useful as standard reference when we face various financial decision-making situations over annuity and lump sum amount. Moving to simplified scenario, there are two options at retirement stage between 100 million Korean Won lump sum and 6 million Korean Won per year. Which option is the favorable one? For man at age 65 with 2% interest rate, the annuity factor is lower than suggested value (15.62<100/6), meaning that lump sum is the favorable choice. On the other hand, for woman at age 65 with 2% interest rate, the annuity factor is higher than suggested value (18.39>100/6), is such case annuity is more favorable. We can choose not only between annuity and lump sum amount at retirement, but also to buy annuity with lump sum amount and even decide the timing of purchase. Depending on the circumstance, we can find the best solution based on APV (Actuarial Present Value) criteria. However, we should bear in mind that there is discrepancy between the generality of statistics and individual cases.

There are two cases where the probability in life table and applied probability for individual decision making is different. First of all, let us think about high health risk. For example, people who regularly involve in risky activities or have bad habits in regards to health are evidently exposed to higher risk. In this context, physical hazard has a direct impact on the survival probability. In this case, we can figure out that applied real annuity value for high risk individual would be lower due to lower survival rate. Second of all, subjective recognition of probability matters because there is often considerable gap between recognized probability and statistics. Kahneman wrote that “Our expectations about the frequency of events are distorted by the prevalence and emotional intensity of the messages to which we are exposed” [Kahneman (2012), p138]. We are inclined to put more weights on the frequency of events through personal experience such as direct experience or information from surroundings. Our perception of probability is unconsciously distorted by media coverage. It overemphasizes the risk of event by repeating intended information. In particular, intense image evokes attention because memory lasts longer and persistently has an impact on awareness. Therefore, prudent consideration is required for decision-making.

Macaulay duration is defined as weighted average time until cash flows are received, and is measured in years (John, 1993). In practice, the index is used for measuring the sensitivity of asset and liability for a unit change in interest rate. Life annuity has a lower Macaulay duration than annuity certain. It means that life annuity has lower weighted average time until cash flows are paid back and less price sensitivity for a unit change in interest rate than annuity certain.

Proof. See Appendix B.

2.4 Exchange Ratio

Let us suppose that there is a person who is 35 years old now. The person will be retired at age 65 and is expected to live until 85 years old. As a reference point, life expectancy at birth in 2017 is 82.7 years and approaches towards 85 years (KOSIS, 2017a). The whole period is divided by following pattern: the period of 30 years will be set as accumulation phase and the period of 20 years as withdrawal phase. The exchange ratio, which is the ratio of consumption amount to saving amount, illustrates how much money from individual income should be saved for retirement to reach the target of living standard [See Milevsky (2006), p37]. In other words, the exchange ratio is the ratio between annual expenditure during withdrawal phase after retirement and annual contribution during accumulation phase before retirement.

$$S\left(s_{\overline{10|}i_1}(1+i_2)^{10}(1+i_3)^{10}+s_{\overline{10|}i_2}(1+i_3)^{10}+s_{\overline{10|}i_3}\right)=C\left(a_{\overline{10|}i_4}+v_{i_4}^{10}a_{\overline{10|}i_5}\right) \qquad (2.4.1)$$

$$a_{\overline{n|}}=v^1+v^2+\cdots+v^n=\frac{(1-v^n)}{i}, \qquad v=\frac{1}{1+i}$$

$$s_{\overline{n|}}=(1+i)^n+(1+i)^{n-1}+\cdots+(1+i)^1=\frac{(1+i)^n-1}{i}$$

where *S* is saving amount, *C* is consumption amount, *C/S* is exchange ratio, i_1 is the interest rate for interval *t=1 to 10*, i_2 is the interest rate for interval *t=11 to 20*, i_3 is the interest rate for interval *t=21 to 30*, i_4 is the interest rate for interval *t=31 to 40*, i_5 is the interest rate for interval *t=41 to 50*. $a_{\overline{n|}}$ is the present value of a certain annuity-immediate with unit amount for *n* periods, $s_{\overline{n|}}$ is the future value of a certain annuity-immediate with unit amount for *n* periods.

The formula (2.4.1) intends to calculate the exchange ratio in 30 years from now. The left side of the formula shows the total accumulated saving amount in 30 years from now ($t=30$) and the right side of the formula shows the total present value of saving amount. The reference point is $t=30$ when the person at current age 35 reaches age 65. The annuity-immediate means that the payment is made at the end of the time interval. The unit time interval in this model is 1 year. Savings account is one of the examples of annuity certain featuring fixed regular benefit with specific term period n. There is implicit assumption that interest rate of every 10 years interval and the amount of saving and consumption for whole period are constant.

Depending on interest rate applied for each 10 years, four scenarios are simulated in the Table 5 below. Since scenario 1 and scenario 4 assume constant interest rate for the whole period, it is possible to observe the impact of interest on exchange ratio. Scenario 2 shows downward slope of interest rate from accumulation phase to withdrawal phase. On the other hand, scenario 3 indicates upward slope of interest rate from accumulation phase to withdrawal phase.

Table 5: Exchange ratio of annuity certain depending on four scenarios

	i_1	i_2	i_3	i_4	i_5	***C/S***
Scenario 1	2%	2%	2%	2%	2%	2.4810
Scenario 2	3%	3%	3%	2%	2%	2.9096
Scenario 3	2%	2%	2%	3%	3%	2.7268
Scenario 4	1%	1%	1%	1%	1%	1.9276

It is obvious that the exchange ratio is higher when the interest rate is high during accumulation phase and high throughout withdrawal phase. For example, generation born in 1960s, so-called baby boom generation, had advantage of historical interest rate movement which is far better than scenario 2 and scenario 3. When it comes to generation born in 1980s and onward, low interest rate was already settled and expected to continue the trend close to scenario 1 and scenario 4. It means that latter generation is required to save more in order to reach the same standard of living of former generation.

According to the scenario 1, if we save 1 Korean Won at the end of each year for 30 years, we will receive 2.4810 Korean Won at the end of year for 20 years after reaching the retirement age of 65. Coming back to the real case in Korea, marriage couple and single person each need to save 11.46 (28.44/2.4810) and 7.01 (17.4/2.4810) million per year Korean Won for 30 years to sustain adequate amount of retirement expenditure for 20 years. When the pension replacement rate in Korea reaches 70%, the required amount of putting aside will be 3.44 and 2.10 million Korean Won per year for 30 years.

Be sure that the bundle of inputs is just one example of many possible scenarios. Exchange ratio is not a fixed number along with timeline. Each individual exchange ratio can be different because of the expectation of future interest rate, the accumulation and withdrawal period under available set of information at present. The structure of model can be adjusted according to various number of payments derived from reasonable assumption based on the information of life expectancy and life cycle. Further consideration of indexes such as economic and demographic outlook is necessary.

$$S\left(s_{35:\overline{10|}_{i_1}}\frac{(1+i_2)^{10}}{p_{45,10}}\frac{(1+i_3)^{10}}{p_{55,10}}+s_{45:\overline{10|}_{i_2}}\frac{(1+i_3)^{10}}{p_{55,10}}+s_{55:\overline{10|}_{i_3}}\right) \tag{2.4.2}$$

$$=C\left(a_{65:\overline{10|}_{i_4}}+v_{i_4}^{10}p_{65,10}a_{75:\overline{10|}_{i_5}}\right)$$

$$a_{x:\overline{10|}}=v^1p_x+\cdots+v^{10}p_{x,10}=\sum_{t=1}^{10}v^tp_{x,t},\quad A_{x:\overline{10|}}{}^1=v^{10}p_{x,10}$$

$$s_{35:\overline{10|}}=\frac{a_{x:\overline{10|}}}{A_{x:\overline{10|}}{}^1},\qquad \frac{1}{A_{x:\overline{10|}}{}^1}=\frac{(1+i)^{10}}{p_{x,10}},\qquad v=\frac{1}{1+i}$$

where $a_{x:\overline{10|}}$ is the present value of an annuity-immediate with unit amount for *10* periods at age *x*, $s_{x:\overline{10|}}$ is the future value of an annuity-immediate with unit amount for *10* periods at age *x*, $A_{x:\overline{10|}}{}^1$ is the present value of pure endowment with unit amount after *10* periods at age *x*. The annuity-immediate indicates that saving and consumption are accumulated at the end of year even though death happens during the year.

Furthermore, what if we consider the factor of mortality rate during this period? Let us design more complicated model by adding new factor of mortality rate as above. The formula (2.4.2) aims to calculate the exchange ratio when the person reaches age 65. The left side of the formula presents the total accumulated saving amount and the right side of the formula presents the total present value of saving amount when the person reaches age 65. One of main examples of annuity is life annuity, whose benefit is paid as long as the recipient is alive. Reflecting the difference of mortality rate between man and woman, following four scenarios are simulated.

Table 6: Exchange ratio of annuity depending on four scenarios for man at *t=30*

Man	i_1	i_2	i_3	i_4	i_5	***C/S***
Scenario 1	2%	2%	2%	2%	2%	3.3591
Scenario 2	3%	3%	3%	2%	2%	3.9490
Scenario 3	2%	2%	2%	3%	3%	3.6529
Scenario 4	1%	1%	1%	1%	1%	2.6320

Table 7: Exchange ratio of annuity depending on four scenarios for woman at *t=30*

Woman	i_1	i_2	i_3	i_4	i_5	*C/S*
Scenario 1	2%	2%	2%	2%	2%	2.8383
Scenario 2	3%	3%	3%	2%	2%	3.3317
Scenario 3	2%	2%	2%	3%	3%	3.1026
Scenario 4	1%	1%	1%	1%	1%	2.2155

It is shown that the exchange ratio is larger when mortality rate is incorporated into the model than annuity certain. Overall, the exchange ratio is around 35% higher for man and 15% higher for woman. Man shows higher exchange ratio than woman as the mortality rate of man is higher than of woman. That is, man enjoys higher standard of living in given saving amount.

The remarkable structural difference between two models is the matter of mortality factor. Considering the trait of life annuity, the contract is closed when a person passes away. The policyholder will get benefit from age 65 as long as being alive. If policyholder at age 35 passes away in accumulation phase, the policyholder will not get benefit and the accumulated saving amount is invalid. It is called cross-subsidy that dead policyholders support living policyholders. Cross-subsidy in man's group happens more frequently. As man takes more mortality risk, he gets compensation with higher exchange ratio. It is easy to infer that the impact of mortality rate on the exchange ratio will be smaller, because mortality improvement will be slower in the future.

According to the scenario 1, if single man saves 1 Korean Won at the end of each year for the period of 30 years, then he will receive 3.3591 Korean Won at the end of year for 20 years after reaching the retirement age of 65. Referring to the survey conducted in Korea, single man and single woman each need to save 5.18 (17.4/3.3591) and 6.13 (17.4/2.8383) million Korean Won per year for 30 years to sustain adequate amount of retirement expenditure for 20 years. In case that the target pension replacement rate in Korea is fulfilled, the required amount of putting aside will be 1.55 and 1.84 million Korean Won per year for 30 years. In addition, it is important to present the 30 years survival rate of man and woman at age 35 i.e. $(p^m_{35,30}, p^w_{35,30})$ = (0.8898, 0.9570), the 20 years survival rate of man and woman at age 65 i.e. $(p^m_{65,20}, p^w_{65,20})$ = (0.4587, 0.6700). After that, if a person survives with probability $(p^m_{35,50}, p^w_{35,50})$= (0.4081, 0.6412) until age 85, the rest of life solely depends on public pension in Korea. In fact, the mortality rate is remarkably steep from this point (the mortality rate of man is over 10% and woman is over 6%), the 5 years survival rate of man and woman at age 85 is $(p^m_{85,5}, p^w_{85,5})$ = (0.5027, 0.6269). Note that *m* is man and *w* is woman.

In one year, the exchange ratio of annuity certain in scenario 1 will be the same if the interest rate is 2%. However, the exchange ratio of annuity in scenario 1 will be higher than current level with 2% interest rate if a person survived.

2.5 *Ruin Probability*

As time goes by, supposing that you are 65 years old now and since retirement life begins, your wealth starts to shrink from this point on. You decide to approach retirement issue with a new strategy. The strategy is that you spend your last money on your last living day, so-called "die broke" strategy [Milevsky (2006), p33]. Under this model, you will face one event in the end: bankruptcy or death. That is, wealth will be depleted while you are alive or taking a last breath while leaving bequest. The ruin probability model shows the chance of case when you finish all you savings before passing away [See Milevsky (2006), p30].

$$\sum_{t=1}^{\infty}(1-p)\binom{t-1}{k}p^{t-1-k}(1-p)^k(1-q)^t = \left(1-\frac{q}{1-p+pq}\right)^{k+1} \qquad (2.5.1)$$

$$\text{Probability of bankruptcy at time } t\text{: } (1-p)\binom{t-1}{k}p^{t-1-k}(1-p)^k, \qquad t = k+1, \cdots$$

where p is coverage ratio, k is the largest integer less than W_0, W_0 is initial budget, q is mortality rate, and t is time. Probability of bankruptcy at time t: $(1-p)\binom{t-1}{k}p^{t-1-k}(1-p)^k$ follows Negative Binomial distribution (k+1, p). E(T-1-K) = $\frac{(k+1)p}{1-p}$, $\mathrm{E(T)} = \frac{k+1}{1-p}$, $\mathrm{Var(T)} = \frac{(k+1)p}{(1-p)^2}$.

Proof. See Appendix C.

The integer k is the integer part of ratio between initial budget W_0 and yearly expenditure. Throughout this model, yearly expenditure is 1 as unit amount. At the age of 65, you have initial budget W_0. After time t, you will have budget W_t. It is said that you are bankrupted when W_t is zero. The relationships of wealth between time t and $t+1$ is $W_{t+1} = W_t+X_t-1$, which indicates that the amount of wealth in unit period moves in two possible ways depending on a random variable X_t. The random variable X_t has binomial distribution with probability p. The probability determines whether the yearly expenditure is financed or not. That is, if X_t is 1, $W_{t+1} = W_t$ with probability p and if X_t is 0, $W_{t+1} = W_t-1$ with probability $1-p$. Hence, one way is that your budget will be the same as your expenditure is financed. The other way is that your budget will go down by a unit as your yearly expenditure is not financed. The process lasts as long as you survive.

Matching the first moment, which is expected value, is reasonable way to estimate the probability of variable X_t. Mathematically, $E(X_t) = 1*p+0*(1-p) = p$. For instance, if the pension is 1/3 of expenditure, then p can be estimated as 1/3 because it takes three periods to pay expenditure on average. As t grows along with time, there are more number of results X_t and the average value of X_t is asymptotically close to p according to Law of Large Numbers (LLN) theorem. Reflecting the feature, let us call p as coverage ratio. In this context, the coverage ratio is the ratio between the annual pension paid by public pension system to annual expenditure after retirement.

In order to understand the logic of estimation on the probability of variable X_t, let us imagine a machine and make an experiment. When you insert a coin, the machine will give you a ball in return. The probability of white ball is 1/3 and black ball is 2/3. If you insert coins until you get a white ball, what is the average number of coins to insert? It is the example of geometric distribution and the answer is 3 ($1/p$, p=1/3). It is reasonable to assume each trial is independent. The logic is aligned with ruin probability example. The probability of white ball is coverage ratio. White ball is analogous to full compensation of expenditure. Coins are like time period. It could be interpreted that it requires 3 periods corresponding to 3 coins to compensate expenditure on average. Even though three trials are needed to get a white ball on average, it is still uncertain until outcome is realized. We can get a white ball in first or second trial but we can get one even at tenth. From the Law of Large Numbers (LLN) theorem, we know that the average outcome is close to prior probability when the number of trial increases. The speed of convergence toward p is dependent on the number of trial and prior probability p.

Using statistics of Korea (KOSIS, 2017a), we derive mortality rate q for the model. We adopt Constant Force of Mortality (CFM) assumption where mortality rate q is required to be constant. Given the data of curtate future lifetime at age 65, we calculate the force of mortality μ and produce mortality rate q in a sequence of step. Before starting, it is worth making clear that the limiting age of life table w becomes infinite here because mortality rate is constant regardless of age. In fact, the limiting age of Korean life table is 100 and naturally p_{99} is 0.

$$\text{Constant Force of Mortality (CFM)}: \mu_x = \mu, \qquad q_x = q \quad \forall x \tag{2.5.2}$$

$$p_x = e^{-\int_0^1 \mu_{x+s} ds} = e^{-\int_0^1 \mu ds} = e^{-\mu} \quad \forall x, \qquad p_x = 1 - q_x$$

$$e_x = p_x + p_{x,2} + \cdots = e^{-\mu} + e^{-2\mu} + \cdots = \frac{1}{e^{\mu} - 1}$$

where e_x is curtate expectation for a life at age x, μ_x is the force of mortality at age x, p_x is survival rate at age x for one year, and q_x is mortality rate at age x for one year.

To begin with, the curtate future lifetime at age 65 of man and woman is 18.6, 22.7 years for each (KOSIS, 2017a). The corresponding force of mortality is 0.0524, 0.0431 for each. Survival rate is 0.9490, 0.9578 for each. Finally, the mortality rate is 0.0510, 0.0422. Note that fixed mortality assumption has some extent of limitation on the interpretation of results. In fact, the derived mortality rate based on Constant Force of Mortality (CFM) assumption overestimates real mortality rate from age 65 to 77 and underestimates from age 78. The impact of gap between Constant Force of Mortality (CFM) assumption and real mortality rate for ruin probability depends on the integer k less than initial budget W_0.

The ruin probability model assumes that the event of bankruptcy or death happens at the end of the time interval. The coverage ratio p and mortality rate q are invariant with time. Taking into account several kinds of the largest integer k less than initial budget W_0 and coverage ratio p, multiple scenarios are simulated in the Table 8 and Table 9. Separate tables of ruin probability are provided because mortality rate between man and woman is different.

It is important to note that the ruin probability is low when the coverage ratio is high or initial wealth at retirement is high. If larger portion of wealth is financed from outside and so considerable wealth is available at retirement, you will be far from bankruptcy and live in decently for the rest of your life. Public pension determines coverage ratio which plays a significant role to lower ruin probability. Looking at the slope of ruin probability in depth, the coverage ratio has more powerful impact than initial wealth. Interestingly, man has lower ruin probability than woman under the same condition. Due to higher mortality rate for all age, man is more inclined to face death rather than bankruptcy. Above “die broke” strategy is viable for man. Like previous exchange ratio model, group of man has more chance to experience cross-subsidy phenomenon. As it seems that the trend of life expectancy continues to grow in the future, ruin probability will be higher given the same coverage ratio and wealth. Therefore, it is necessary to raise coverage ratio by public pension on the national level or wealth on the individual level in order to maintain current level of ruin probability.

Table 8: Ruin probability in relation to coverage ratio and wealth for man

p/k	**4**	**5**	**10**	**14**	**15**
0.4	0.7095	0.6511	0.4239	0.3008	0.2760
0.5	0.6646	0.6001	0.3601	0.2394	0.2161
0.6	0.6038	0.5323	0.2833	0.1711	0.1508
0.7	0.5172	0.4386	0.1923	0.0995	0.0844
0.8	0.3858	0.3041	0.0925	0.0357	0.0281

Table 9: Ruin probability in relation to coverage ratio and wealth for woman

p/k	**4**	**5**	**10**	**14**	**15**
0.4	0.7532	0.7017	0.4924	0.3709	0.3455
0.5	0.7134	0.6556	0.4298	0.3066	0.2818
0.6	0.6584	0.5931	0.3518	0.2316	0.2086
0.7	0.5781	0.5041	0.2541	0.1469	0.1281
0.8	0.4510	0.3696	0.1366	0.0616	0.0505

This model indicates that it is not necessary for individual to save too much to sacrifice present expenditure. Individual can find the best solution based on different respective circumstances. Based on attitude towards risk and expectation of future coverage ratio under available set of information at present, one is viable to set up strategy that ruin probability is less than specific threshold and designs the target of wealth at age 65 accordingly. Furthermore, we can use new inputs after due consideration of changing situation in future.

In Table 8 and Table 9, coverage ratio is presented from 40% to 80% reflecting the situation in Korea. Looking ahead, pension benefits will remain low as the targeted replacement rate for a person with 40 years of contributions was reduced from an initial 70% to 50% in 2007, and is set to decline to 40% by 2028 [S. Jones and Urasawa (2017), p13]. Replacement rate above 40% is fulfilled by supplementary source such as occupational pension.

For instance, if man at age 65 accumulated wealth required to survive around 15 years and the coverage ratio is 40%, the ruin probability is 0.2760. It means that the man will face the event that wealth is depleted before passing away with probability of 27.6%. Now, the goal of coverage ratio is 70% and initial wealth at age 65 is 15 times of yearly expenditure. Then, the ruin probability of man at age 65 is 0.0844 and woman is 0.1281. With the same assumptions, single person needs to prepare 8.7 (17.4*15/30, interest rate is 0%) million Korean Won per year for 30 years. What is the change of ruin probability for man after one year? If wealth reduced by 1 unit, the ruin probability will be 0.3008 with k=14. If wealth is the same as before, the ruin probability will amount to 0.2760 with k=15.

3. National Financial Decision-making on Retirement

3.1 Introduction-illustration of Republic of KOREA

Social security is widely recognized as a primary tool in reducing poverty and preventing vulnerability throughout individuals' life cycle. For more than a century, well-governed social security reduced social contradictions, enhanced national identity, compensated for market failures and contributed to stabilizing, often even stimulating economic growth [Gongcheng and Scholz (2019), p11].

In this sense, the next step is to find the way how to implement well-governed social security system. Solid and rigorous analysis based on mathematics and statistics is necessary especially for social security system which features long-lasting timeline and complicated dynamics among socioeconomics variables. ILO (International Labor Organization) emphasizes that the financial management of a pension scheme on the basis of a sound long-term financial perspective is crucial for ensuring its ongoing sustainability [ILO (1997), p9]. In light of this, actuarial analysis on a national social security pension scheme should lay the cornerstone of social security system.

ILO (International Labor Organization) internal guideline defines that actuarial valuations are intended to review the present and expected future financial developments of existing or new social security schemes with the possibility to include analyses of the financial effects of major structural reforms in the case of existing schemes [ILO (1998), p5]. Actuarial projection requires roughly three assumptions such as population, economic, and policy system. Among them, population assumption has the largest room to utilize actuarial model used past data which significantly enhances the predictability.

The main purpose of this chapter is to present simple and effective models predicting macro-level statistics. Those models can be applied for estimating life table, life expectancy at age 65, APV (Actuarial Present Value) of pension, replacement ratio, trend of newborns, whole population and dependency ratio. Simple and credible estimation and projection technique is vital for policymaker to minimize policy lag from designing to implementation.

3.2 Projection on Mortality Rate and Life Expectancy at age 65 reflecting Mortality Improvement

Throughout decades around the world, the overall mortality rate has kept decreasing because of advanced medical technology, invention of pharmaceuticals and enhanced nutrition treatment. In particular, the data on newborn babies and elderly shows a significant progress of lowering mortality rate. It is evident that the mortality rate is quite different at age 65 in 1970 and in 2000.

$$q_x^{z2} = q_x^{z1} e^{-\alpha_{(z2-z1)}}, \qquad z2 > z1 \tag{3.2.1}$$

where q_x is mortality rate at age x for one year, *z1* refers to base year, *z2* refers to respective years from base year onward, *z2- z1* refers to the number of years from base year, and $e^{-\alpha_{(z2-z1)}}$ is the mortality improvement coefficient such that $e^{-\alpha_1}$, $e^{-\alpha_2}$ for year 1971, 1972 respectively.

The formula (3.2.1) assumes that mortality improvement $e^{-\alpha_{(z2-z1)}}$ is exponential from base year. The process of estimation is as follows. We transform from curtate life expectancy at age 65 data from year 1970 to year 2016 in Korea (OECD, 2019) into the force of mortality with Constant Force of Mortality (CFM) assumption. After deriving mortality rate q each year from 1970 to 2016, we set *z1* as 1970 and *z2* starts from 1971 to 2016. Finally, we implement single linear regression analysis to determine alpha α, which is the coefficient of mortality improvement, along with the number of year from 1970. R program is implemented by OLS (Ordinary Least Squares) method for estimating the coefficient of mortality improvement.

Reference. See Appendix D.

$$\text{Simple Linear Regression: } \hat{y} = \widehat{\beta_0} + \widehat{\beta_1} x, \tag{3.2.2}$$

$$\text{Residual Sum of Squares: } RSS = \left(y - \widehat{\beta_0} + \widehat{\beta_1} x_1\right)^2 + \cdots + (y - \widehat{\beta_0} + \widehat{\beta_1} x_n)^2$$

$$\widehat{\beta_1} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})\,(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \qquad \widehat{\beta_0} = \bar{y} - \widehat{\beta_1}\bar{x}, \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}, \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}, \qquad \text{Total Sum of Squares: } TSS = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$\text{Null hypothesis } H_0: \beta_k = 0, \qquad \text{Alternative hypothesis } H_a: \beta_k \neq 0, \qquad k = 0,1$$

where $\hat{y}$ indicates a prediction of *Y* on the basis of *X*=*x*, $\widehat{\beta_0}$ is intercept, $\widehat{\beta_1}$ is slope, *x* is time, and *n* is the number of data. We use a hat symbol ^, to denote the estimated value for an unknown parameter or coefficient. We predict a quantitative response *y*, which is alpha α_x, on the basis of single predictor variable *x*, which is time. Minimizing least squares approach chooses $\widehat{\beta_0}$, $\widehat{\beta_1}$ to minimize the Residual Sum of Squares (RSS).

Considering Multiple R-squared (R^2= 0.9868 for man and R^2= 0.9603 for woman are close to 1) in Appendix D, the simple linear regression line fits well the data. That is, the deviation from the mean is largely explained by linear regression equation. As the coefficients of (Intercept) and Time are quite big compared to standard errors, the absolute t-values are also very high and p-values are almost 0. Therefore, both coefficients reject the null hypothesis with less than 0.1% significance level and are remarkably different from 0. Because the number of data is 46 and we estimate 2 coefficients, the degree of freedom is 44 (=46-2). Since the coefficient β_1 is slope, the quantitative response y is increased by the amount with the change of time unit predictor variable. Regression line for man shows steeper slope than for woman. Because the data (OECD, 2019) from 1970 to 1980 indicates very slow upward trend of life expectancy, a negative intercept is possible.

From the result of regression presented in Appendix D, the estimate of mortality rate for man at age 65 in 2025 is equal to the mortality rate at age 65 in 1970 base year multiplied by the factor of mortality improvement, in mathematical terms $q_{65}^{z2} = q_{65}^{z1} e^{-\alpha_{(2025-1970)}} = q_{65}^{z1} e^{-\alpha_{55}} = q_{65}^{z1} e^{-(-0.0680+0.0131*55)} = q_{65}^{z1} * 0.6505 = 0.0487$.

Table 10: Mortality Rate at age 65 for man and woman from year 2025 to year 2045 reflecting mortality improvement

	2025	**2030**	**2035**	**2040**	**2045**
Mortality Rate (Man, age 65)	0.0487	0.0456	0.0428	0.0401	0.0375
Mortality Rate (Woman, age 65)	0.0417	0.0398	0.0381	0.0364	0.0348

Table 11: Life Expectancy at age 65 for man and woman from year 2025 to year 2045 reflecting mortality improvement

	2025	**2030**	**2035**	**2040**	**2045**
Life Expectancy (Man, age 65)	19.5	20.9	22.4	24.0	25.6
Life Expectancy (Woman, age 65)	23.0	24.1	25.3	26.5	27.7

Referring to calculation in Table 11, once Korean man reaches age 65 in 2045, man is expected to live on average 25.6 years more until age 90.6. As for woman with the same condition, woman will live on average 27.7 years more until 92.7. The gap of both indexes between man and woman will become narrow in the future. When it comes to life expectancy, the gap in 2025 is 3.5 years and is expected to drop to 2.1 years in 2045. In Table 10, the gap of mortality rate between man and woman in 2025 is 0.007 and is expected to be 0.0027 in 2045. Model shows that survival rate will improve by more than 1% for man and less than 1% for woman in the period of 25 years.

3.3 Projection on Population and Dependency Ratio-illustration of Republic of KOREA

Projection on population has a pivotal role for actuarial valuation of financial status on public pension because it is fundamental component to assess the number of member and pensioner in future periods. It determines the trend of expenditure of pension and the plan of fund provisions. Furthermore, the structure of population directly affects staple macro-economic variable such as growth rate of economy, the rate of wage rise, interest rate, and inflation.

$$l_0^{y+1} = \sum_{x=15}^{49} l_x^{w,y} * f_x^{w,y} \tag{3.3.1}$$

where l_0^{y+1} is the number of baby in year *y+1*, $l_x^{w,y}$ is the number of woman at age *x* in year *y*, $f_x^{w,y}$ is the fertility rate of woman at age *x* in year *y*, *y* is year from 2018, 2019, and so on, *x* is woman's age from 15 to 49 regarded as childbearing age.

In formula (3.3.1), age-specific fertility rate statistics (KOSIS, 2017b) is referred. Age-specific fertility rate is defined as the number of baby divided by the number of woman (KOSIS, 2017b). Therefore, the number of baby in year *y+1* is calculated as total summation of the number of woman multiplied by corresponding fertility rate at age *x* in year *y*.

To count the number of woman at age *x* in 2017 is complex due to the fact that data availability is limited. As the population data in Korea is presented as 5-year unit basis, it is necessary to extract age-specific population from integrated value in 5-year bracket.

$$l_x + l_{x+1} + l_{x+2} + l_{x+3} + l_{x+4} = l_x + l_x p_{x,1} + l_x p_{x,2} + l_x p_{x,3} + l_x p_{x,4} \tag{3.3.2}$$
$$x = 0,5,\cdots,95$$

where l_x is the number of persons who attain age *x* according to mortality table [Perryman (1950), p125], $p_{x,t}$ is conditional probability of an *x*-year-old surviving *t* more years.

Through the equation, we can calculate the number of persons in the first year of 5-year bracket with given total mortality rate data. For example, the number of people from age 55 to age 60 is 4,258,232 (KOSIS, 2017c). In this case, the number of people at age 55 is 857,848. The next step is to find the number of woman for all age.

$$l_x = l_x^m + l_x^w, \qquad l_{x+1} = l_x^m p_x^m + l_x^w p_x^w \tag{3.3.3}$$
$$\therefore p_x = \frac{l_{x+1}}{l_x} = a p_x^m + (1-a) p_x^w, \qquad \frac{l_x^m}{l_x} = a$$

where *m* is man, *w* is woman, and *a* is the proportion of man at age *x*.

Based on given total population, man and woman mortality rate data, the proportion of man is derived. The newborn babies each year are included in whole population and follow current mortality rate. As fertility rate is applied for woman from childbearing age, the additional step is to count the number of daughters each year. Subsequently, it requires male to female ratio at birth, which in Korea amounted to 116.5 in 1990 and 110.1 in 2000 (KOSIS, 2017d). In the 1st actuarial projection, the male to female ratio at birth assumption for actuarial projection is 106.2 onward 2020 [NPS (2003), p56]. In the 2nd actuarial projection, the ratio is 106.0 onward 2026 [NPS (2008), p59]. In the 3rd actuarial projection, the ratio is 105.4 onward 2025 [NPS (2013), p63]. In the 4th actuarial projection, the assumption is that the ratio keeps the average of recent 3 years, which is around 105 [NPS (2018), p49]. Taking the latest assumption, population projection is implemented.

Last process for estimating population is the adjustment of net international migration rate in Table 12. Here the 4th assumption is used as described above for male to female ratio at birth.

Table 12: Illustration on net international migration rate assumption from 2020 to 2060 in Republic of KOREA (Unit: per 1000 persons)

International Migration	***2020***	***2030***	***2040***	***2050***	***2060***
4th Actuarial Projection	1.20	0.60	0.70	0.70	0.70
3rd Actuarial Projection	0.71	0.57	0.63	0.64	0.53

Source: National Pension Service [NPS (2013), p65| (2018), p51]

The following paragraphs will be discussing the results of four actuarial projections done by National Pension Service (NPS) and New actuarial projection calculated using above described method.

Table 13: Illustration on actuarial projections of population by estimation and reference from year 2020 to 2045 in Republic of KOREA (Unit: 1000 people)

Population	***2025***	***2030***	***2035***	***2040***	***2045***
New Actuarial Projection	52,103	52,079	51,438	50,187	48,363
4th Actuarial Projection	52,610	52,941	52,834	52,198	51,051
3rd Actuarial Projection	51,972	52,160	51,888	51,091	49,810
2nd Actuarial Projection	49,108	48,635	47,734	46,343	44,521
1st Actuarial Projection	50,649	50,296	49,484	48,204	46,471

Source: National Pension Service [NPS (2003), p67| (2008), p74| (2013), p78| (2018), p70]

Table 13 presents the results of population projection. In New actuarial projection mortality improvement relevant to cohort is reflected to estimation. According to New actuarial projection, the peak of population is 52,125 (Unit: 1000 people) in 2027. The structure of population will be matured from 2060 based on the slope of population decrease. The change of population is -1.5% in 2060 and the increment is around -0.005% (5 people per 100000 people) onward. In order to maintain the slope of population decrease to the level in 2060, policymaker should lift the net international migration rate from 0.70 to 0.75 (from 70 people to 75 people per 100000 people) by adjusting existing quota system. However, if the target of sustaining the population is urgent, then it would be necessary to increase net international migration rate from 0.70 to 15.70 (from 70 people to 1570 people per 100000 people) by designing new immigration system.

Table 14: Illustration on actuarial projections of dependency ratio by estimation and reference from year 2020 to 2045 in Republic of KOREA (Unit: %)

Dependency Ratio	*2025*	*2030*	*2035*	*2040*	*2045*
New Actuarial Projection	31.6	40.2	49.5	58.8	66.8
4th Actuarial Projection	30.6	39.8	49.8	60.7	68.6
3rd Actuarial Projection	30.8	40.2	49.9	59.8	67.3
2nd Actuarial Projection	30.2	39.2	48.6	59.1	67.3
1st Actuarial Projection	29.5	37.3	45.3	54.0	59.9

Source: National Pension Service [NPS (2003), p67| (2008), p74| (2013), p78| (2018), p70]

Note that the definition of dependency ratio in pension is slightly different from the OECD definition (See page 2). Here dependency ratio is the number of individuals aged 65 and over per 100 people of those aged between 18 and 64 (OECD, 20 and 64). During 20 years the dependency ratio increases more than 2 times as presented in Table 14. The structure of population faces revolutionary reform, which is unprecedented event in the history of Korea. The structure of population will be stable from 2063 based on the slope of dependency ratio increase. The change of dependency ratio is less than 1% in 2063 onward.

Behind the data in Table 15, there is hidden complicatedly tangled context. Baseline Total Fertility Rate (TFR) for New actuarial projection has been selected 3 years before valuation year. For example, the 1st actuarial projection regards baseline Total Fertility Rate (TFR) in 2000, which is 1.47 [NPS (2003), p55]. In the 2nd actuarial projection, the rate fell to 1.08 in 2005 [NPS (2008), p56]. The change of real Total Fertility Rate (TFR) was so drastic from 2000 to 2005 that the assumption became volatile. The real rate was at the bottom in 2005 and increased from 2005 to 2010. In the 3rd actuarial projection, the rate is 1.23 in 2010 [NPS

(2013), p63]. The rapid up and down from 2000 to 2010 causes roller-coaster assumption. In fact, it is difficult to project future Total Fertility Rate (TFR) properly as there is no clear correlation year by year from sharp fluctuation over 10 years.

Table 15: Illustration on Total Fertility Rate (TFR) assumption from 2020 to 2040 in Republic of KOREA (Unit: per woman)

Total Fertility Rate	***2020~***	***2025~***	***2030~***	***2035~***	***2040~***
New Actuarial Projection	0.90	1.00	1.14	1.22	1.27
4th Actuarial Projection	1.24	1.28	1.32	1.36	1.37
3rd Actuarial Projection	1.35	1.38	1.41	1.41	1.42
2nd Actuarial Projection	1.20	1.25	1.28	1.28	1.28
1st Actuarial Projection	1.37	1.38	1.39	1.40	1.40

Source: Korean Statistical Information Service (KOSIS, 2017b),
National Pension Service [NPS (2003), p55| (2008), p56| (2013), p63| (2018), p48]

Moreover, there is implicitly interesting context in the 4th actuarial projection. The baseline rate Total Fertility Rate (TFR) amounted to 1.24 in 2015 [NPS (2018), p48]. In NPS (National Pension Service) report it is even written that the very low rate of 1.05 in 2017 is reviewed as the baseline rate. However, the new information from available statistics seems to not have affected the assumption of the future Total Fertility Rate (TFR). The quite low rate in 2017 might be regarded as a temporary downturn, therefore baseline rate in 2015 was chosen. It turned out that the shift of low rate was not temporary, because in 2018 it resulted in 0.977. KOSIS (Korean Statistical Information Service) set Total Fertility Rate (TFR) assumption reflecting latest available statistics in 2017, which was used in New actuarial projection. The rate is at the lowest level of 0.86 in 2021, slowly goes up and becomes stable at the level of 1.27 from 2040 on. In Table 15 the future projection shows quite a big difference of 0.34 in 2020 and convergence to 0.1 in 2040. That is the underlying reason of the difference between New actuarial projection and 4th actuarial projection.

As Total Fertility Rate (TFR) is determined by socio-economic circumstance and normative family value in society, it shapes slowly and lasts long. Nevertheless, the assumption of it has been unstable in four actuarial projections. It loses the credibility of population projection which is the foundation of designing policy. Therefore, it is important to keep the assumption to be prudent and consistent. Moving on, we demonstrate that Total Fertility Rate (TFR) assumption has a crucial role to explain the estimates of population and demographic old-age dependency ratio.

It is obvious that the first and foremost factor on population is Total Fertility Rate (TFR). It has a sweeping impact on demographic structure over inter-generation period. In order to illustrate the impact of Total Fertility Rate (TFR), population and demographic old-age dependency ratio are measured by changing Total Fertility Rate (TFR) from 1.30 to 2.10 by 0.01 unit from 2025 onward. The selected results will be presented in Table 16. Technically, the age-specific fertility rate to Total Fertility Rate (TFR) is proportionally reflected in simulation. As long as dependency ratio increases, ageing transition of population keeps going. As Korea has the peak of dependency ratio in 2067, the burden of working population grows substantially. Once the Total Fertility Rate (TFR) is above specific threshold (1.47), then peak of dependency ratio happens already in 2062 which is 37 years after new rate application. In conclusion, it takes a lot of time to see the visible impact on the structure of population. Also, we can infer that it is very difficult to turn around the current structure of population.

Mathematically, inflection point is defined as the curves change from concave to convex or vice versa. Since the population curve goes down as time goes by, in such case we speak of change from concave to convex. At this point, the change of slope, i.e. second derivative is from negative to positive the absolute value of slope, i.e. first derivative is the lowest. Inflection point is important barometer because it indicates the maturity of population. Since Korea has no inflection point by 2067 in New actuarial projection, population is not matured yet. It means that the considerable decline of population lasts long and the shape of population is still imbalanced. Likewise once the Total Fertility Rate (TFR) is above specific threshold (1.76), inflection point exists around 2055 when 30 years pass after new rate application in 2025. The higher the rate is, the earlier the inflection point occurs with slower pace of decline.

Table 16: Illustration on actuarial projections of population in 2067, peak dependency ratio, inflection point of population with Total Fertility Rate (TFR) (Projection period: 2017~2067) (Population Unit: 1000 people, Dependency Ratio Unit: %, Year)

TFR	***Population in 2067***	***Dependency Ratio***	***Inflection Point***
1.30	37,714	78.9 (2067)	41,813 (-1.43%, 2059)
1.45	39,480	74.8 (2067)	44,156 (-1.28%, 2057)
1.48	39,840	74.1 (2062)	44,404 (-1.26%, 2057)
1.58	41,056	72.1 (2062)	45,721 (-1.18%, 2056)
1.77	43,435	68.7 (2062)	47,698 (-1.04%, 2055)
1.90	45,115	66.6 (2047)	48,716 (-0.95%, 2055)
2.02	46,703	66.2 (2047)	50,047 (-0.88%, 2054)

$$\text{Population in 2067: } \widehat{y_{population}} = 37457 + 125.8x_{increment}, \qquad R^2 \approx 1 \qquad (3.3.4)$$
$$\text{Peak Dependency Ratio: } \widehat{y_{ratio}} = 77.41 - 0.165x_{increment}, \qquad R^2 \approx 0.96$$

To compute the results presented in Table 16, VBA (Visual Basic for Applications) program in Excel spreadsheet is used for the illustration as 81 (From 1.30 to 2.10 both inclusive with unit 0.01) repetitive inputs are required. As mentioned already, Total Fertility Rate (TFR) is applied as a unit rate from 2025 onward. Total Fertility Rate (TFR) points are selected whenever peak dependency ratio of year or inflection point of year changes.

Regarding dependency ratio, the first threshold of Total Fertility Rate (TFR) is 1.48 when the year of peak is changed from 2067 to 2062. After that, the next transition happens when the rate is 1.90, causing that the peak dependency ratio is shortened by 15 years from 2062 to 2047. Even though the impact of the rate is limited under current structure of population, policymaker should maintain consistent population policy in a long term. When it comes to the population in 2067, the percentage of decline seems small but even 0.01% (0.0001) of fifty million is equal to 5,000 people.

Let us show numerical example. When Total Fertility Rate (TFR) is 1.7, the population in 2067 is 37,457+125.8*40=42,849 (Unit: 1000 people). Likewise, the peak dependency ratio is 77.41-0.165*40=70.81 (Unit: %). As the unit of Total Fertility Rate (TFR) change is 0.01, the increment is (1.7-1.3)*100=40. These two equations imply that population in 2067 will increase by 125,800 people and the peak dependency ratio will decrease by 0.165% point along with Total Fertility Rate (TFR) increase by 0.01 within the boundary of 1.3 to 2.1. From the results, it is not hard to imagine that the level of population and dependency ratio is not sustainable within decades if the current demographic situation lasts.

From above illustration, we acknowledge that it is necessary to smooth the demographic cliff. It suggests that government makes concerted efforts to promote the environment for woman to consider having more children. If the actions to meet the target Total Fertility Rate (TFR) are late, the cost borne by society will tremendously increase.

Various population policies such as inflow of immigrant, paternity and mortality leave, etc. should be timely implemented in order to make better environment for bringing up children and lessen the drastic hit of population. The underlying matter is to deal with price of housing, standardization of education as well as strengthening human labor right, etc. for promoting the standard of living. In the long term, the extensive reform of labor structure will be inevitable in the following years. However, there is no right answer which one should be a priority. Also, even the target combination of dependency ratio and population is normative. Those matters are the realm of social consensus through consulting process among various stakeholders.

3.4 Actuarial Valuation and Sustainability of Public Pension-illustration of Republic of KOREA

With our best possible estimation of population, follow-up procedure is to evaluate the stability and sustainability of pension scheme and analyze actuarial gains and losses between actuarial assumptions and actual experience. Financial estimation model, as an actuarial model, is designed to embody the detailed contents of pension system in mathematical terms. It is comprised of assumption inputs, estimation of demographics and financial estimation in this particular order. Note that every step of calculation is successively done in a series of sequence. The result from previous stage will be input for next one. Assumption inputs part is broadly classified into macro-economic and scheme-related variables. Estimation of demographics forecasts the number of members and pensioners, which is one of inputs for financial estimation. Finally, financial estimation is comprised of outputs such as the inflow of pension premium, outflow of pension benefit, return of investment and accumulated fund level to evaluate stability and sustainability of pension scheme.

The formula of basic pension in Korea has been slightly modified during the last 20 years. When the 4th actuarial projection is implemented, the latest one is as below.

$$\text{Pension: } P = \frac{1}{d}\big(1 + 0.05(d - 20)\big)\sum_t c_t(a_t A + b_t B)I_t\,, \qquad d = \sum_t I_t \geq 10 \tag{3.4.1}$$

where A is the average income of participants per month, B is the average life income of individual participant per month and past income is revaluated based on the growth rate of A value for calculation, c_t is the coefficient of benefit and values are as follows: 2.4 from 1988 to 1998, 1.8 from 1999 to 2007, 1.5 in 2008 and decreases by 0.015 per year until 1.2 in 2028, a_t and b_t are the coefficient of determination which affects the degree of income redistribution and values are as follows: a_t is 1 and b_t is 0.75 from 1988 to 1998 and both are 1 in 1999 onward, I_t is indicator variable which is 1 when premium is paid and 0 when otherwise, and d is the number of participation period. The formula indicates that the basic participation period is 20 years and the pension increase 5% by unit year of participation [See NPS (2018), p125].

From the formula, several features of public pension contrary to private pension can be uniquely inferred. Public pension plays an essential role of income redistribution and subsidy, which contributes to lessen income inequality and lower poverty rate in the long run. That is, dignified standard of living for rest of one's life is promoted for members in society through public policy. Wealth is effectively transferred from high income bracket to low income bracket by means of pension. In fact, the function toward the goal of income equality became weaker than before because the weight of coefficient a_t was 4/7 (1/1.75) from 1988 to 1998 and 1/2 in 1999 onward. When it comes to subsidy, the public pension is more generous as the coefficient of benefit c_t is higher. Therefore, the advantage of pension has been significantly undermined in a short history of public pension. When c_t is 1.2 in 2028 onward, it determines the replacement rate 20% (40%) for the participant who has average income and 20 (40) year's participation period basis. Surprisingly, from 1988 to 1998 the replacement rate was 40% (80%) for the participant who has average income and 20 (40) year's participation period basis. At the very introductory phase of the public pension policy, it is necessary to offer an attractive option for stable establishment of new system. Not long after introduction of policy, it becomes obvious that the favorable option is not sustainable in the environment of low growth rate and drastic demographic transition. By a rule of thumb, the replacement rate for the participant who has average income and 20 year's participation period is equal to the coefficient c_t divided by 6 in current settings. As the increase of pension is 5% constant flat rate per unit year of participation, there is actually no incentive to participate in public pension policy for longer period.

Table 17: Illustration on replacement ratio with five income brackets for 20 year's participation period basis depending on the coefficients *(a_t, b_t)* (Unit: %)

B/(a_t, b_t)	***(1,1)***	***(1,3/4)***	***(8/7,6/7)***	***(7/8,7/8)***	***(5/4,3/4)***	***(3/2,1/2)***
1,220,000	30.7%	28.1%	32.1%	26.8%	33.2%	35.8%
1,830,000	23.9%	21.3%	24.3%	20.9%	24.7%	25.6%
2,440,000	20.4%	17.9%	20.4%	17.9%	20.4%	20.4%
3,050,000	18.4%	15.9%	18.1%	16.1%	17.9%	17.4%
3,660,000	17.1%	14.5%	16.6%	14.9%	16.2%	15.4%

The above list in Table 17 comprising of five income brackets ranges from 50% to 150% with 25% interval, the average income being 2,440,000 Korean Won. The coefficient of benefit c_t is exactly proportional to replacement ratio for every income bracket so this variable is excluded for analysis. As shown above, two coefficients of the first column is current basis from 1999 and second column is from 1988 to 1998. Because the sum of two coefficients is from 2 to 1.75, the replacement ratio is also proportionally from 20.4% to 17.9% (20.4*87.5%). In reality, the replacement ratio of standard case from 1998 to 1999 is from 35% to 30% (35*(1.8/2.4)*(8/7)) as there is trade-off of decreasing c and increasing the sum of a_t, b_t. The third column was determined by using the proportional ratio as in the second column and at the same time matching the sum of two coefficients from the first column. The fourth column is the other way around. Analogously, c_t and the sum of a_t, b_t are related to the size of pie and the ratio a_t to b_t is the matter of how to divide the pie. The conclusion that can be drawn from looking at the Table 17, is that the income redistribution effect gains momentum when the coefficient a_t occupies big proportion. Last two columns are introduced since it is good standard that the 50% income bracket of average has replacement ratio at least two times more than the 150% income bracket of average by actuarial judgment. Then, three times gap of income will be 1.5 times gap of benefit. If Korean society pursues the value of equality and social integrity by reducing poverty, the ratio a_t to b_t should be based on last two columns. It is the way to solve outstandingly high poverty rate at age 65.

In spite of actual coefficients (the ratio a_t to b_t is (1,1)), current pension system is still favorable for participants. The statement can be verified by comparing actuarial contribution rate with actual contribution rate. In order to assess appropriate contribution rate, a formula (3.4.2) based on actuarial equivalence principle is devised.

$$\text{APV(Premium)}: r(12\text{B})\ddot{a}_{65-d:\overline{\text{d}}|}, \qquad d = \sum_t I_t \geq 10, \qquad v = \frac{1}{1+i} \tag{3.4.2}$$

$$\text{APV(Benefit)}: v^d p_{65-d,d}\ddot{a}_{65}\frac{12}{d}\big(1 + 0.05(d-20)\big)\sum_t c_t(a_t A + b_t B)I_t$$

$$+r(12\text{B})\left(\ddot{a}_{\overline{1}|}vq_{65-d} + \ddot{a}_{\overline{2}|}v^2 p_{65-d}q_{65-(d-1)} + \cdots + \ddot{a}_{\overline{\text{d}}|}v^d p_{65-d,d-1}q_{65-\{d-(d-1)\}}\right)$$

$$\therefore r = \frac{v^d \ddot{a}_{65}P}{B\ddot{a}_{\overline{\text{d}}|}}$$

where *r* is contribution rate, $\ddot{a}_{65-d:\overline{\text{d}}|}$ is the present value of an annuity-due with unit amount for *d* periods at age *65-d*, $p_{65-d,d} = \prod_{j=0}^{d-1} p_{65-d+j}$ is conditional probability of an *(65-d)*-year-old surviving *d* more years, $\ddot{a}_{65}$ is the present value of an annuity-due with unit amount for whole period at age *65*, $\ddot{a}_{\overline{\text{d}}|}$ is the present value of an certain annuity-due with unit amount for *d* periods, q_{65-d} is mortality rate at age *65-d* for one year, *d* is the number of participation period, and *P* is the fixed monthly amount of pension.

Proof. See Appendix E.

The description of formula is as follows: Individual participant pays premium at the beginning of each year for *d* period from age *65-d* until age 64 without period of inactivity. The individual participant pays the amount of annual average life income multiplying contribution rate, i.e. *r*(12B)*. The annuity starts at the beginning of each year from age *65* with amount of annual pension, i.e. *(12P)*. If the individual participant passes away before reaching age 65, accumulated premium with accrued interest will be returned at the end of death year as lump sum refund. In terms of actuarial equivalence, contribution rate *r* makes APV (Actuarial Present Value) of premium and benefit equal.

Table 18: Illustration on contribution rate by actuarial equivalence principle depends on participation period and interest rate (B is 2,440,000, the coefficients a_t and b_t are 1)

d/i	***0.0%***	***0.5%***	***1.0%***	***1.5%***	***2.0%***	***2.5%***
10	22.2%	20.4%	18.7%	17.3%	16.0%	14.8%
15	21.9%	19.8%	18.0%	16.4%	14.9%	13.6%
20	21.7%	19.4%	17.4%	15.7%	14.1%	12.7%
25	21.6%	19.1%	16.9%	15.0%	13.3%	11.8%
30	21.5%	18.8%	16.4%	14.4%	12.5%	11.0%

Table 18 illustrates that the contribution rate is inclined to go up with shorter participation period and lower interest rate. When interest rate is low, the difference in contribution rate decreases. The contribution rate depending on participation period indicates that it is advantageous to those who have shorter participation period. The 5% flat rate increase of pension should be reconsidered in order to lead longer participation period. Generally, the change of contribution rate is quite sensitive to the change of interest rate. When participation period is 20, the difference of contribution rate is 1.4% when interest rate changes from 2.5% to 2.0% but it becomes 2.3% when the interest rate changes from 0.5% to 0.0%.

When the participant on standard average income bears around 14% of contribution rate (2% interest rate, 20 years participation period), the break-even of the individual balance can be achieved as the Table 18 shows. Surprisingly, the required contribution rate in practice would be higher considering other benefits such as survivor's pension, disability pension, etc. Even if the calculated contribution rate 14% is quite higher than current rate 9%, this contribution rate 9% has not been changed for last 20 years. The fact implicitly infers that the matter of public pension is very sensitive issue in politics so decision-making carries political burden. The impact of policy on public pension is not clearly visible within decades. Moreover, the person who takes initiative to implement new policy and the person who eventually would take responsibility are inevitably different people. Therefore, consultation of intergeneration based on social integrity and trust is desirable. As the structure of population will put pressure on the sustainability of pension, the potential deficit will grow into a heavy burden in the future. Therefore, all-round measures should take initiative in due course.

Table 19: Illustration on IRR (Internal Rate of Return) depends on participation period (B is 2,440,000, the coefficients a_t and b_t are 1, and contribution rate is 9%)

B/d	***10***	***15***	***20***	***25***	***30***
2,440,000	5.91%	4.86%	4.16%	3.64%	3.24%

The formula (3.4.2) illustrates that the negative cash flows appear as contribution for participation period and positive cash flows appear as pension from age 65. Technically, IRR (Internal Rate of Return) makes the APV (Actuarial Present Value) of cash flows 0. For instance, as outlined in Table 19 a participant who has average income of 2,440,000 Korean Won per month with 20 year's participation period yields 4.16% of return per year. As the participation period decreases, the series of cash flows becomes more lucrative. In practice, public pension tracks the record of individual's contribution to reflect the positive relationship between participation period and income [NPS (2018), p127]. Hence, in reality the difference of IRR (Internal Rate of Return) among participation periods is smaller. It is reconfirmed that 3.24% IRR (Internal Rate of Return) is surely higher than 2.5% interest rate when the participation period is 30 because the required contribution rate 11% by actuarial equivalence principle is above current level of 9%. Considering low interest rate is currently prevalent, public pension at present actually offers favorable investment return for the participants.

$$(12P_x)\ddot{a}_x = (12P_{65})\ddot{a}_{65}, \qquad x = 60,61,\cdots,65,\cdots,69,70 \tag{3.4.3}$$

where P_x is the fixed monthly amount of pension at age *x*.

When it comes to early or late pension option, the appropriate amount of pension should follow actuarial equivalence principle above. Currently, the policy that for early pension option with 0.5% reduction per month and for late pension option with 0.6% increase per month is effective [NPS (2018), p175-185]. Individual decision-making regarding of the option rests on subjective life expectancy. For example, participant who belongs to high income bracket tends to make use of late pension option, which causes additional burden for public pension. To lessen the adverse selection, the alternative way is to develop adequate mortality rates for each income bracket instead of one unit rate. Regarding the alternative way, if the high income bracket shows high survival rate, the percentage increase for late pension option becomes smaller because high survival rate increases the annuity factor. Estimating mortality rate on corresponding income bracket, the records of early or late pension option application along with income bracket can be used as sample data.

Financial estimation of public pension in macro-level begins with estimating each component of the equation below. The accumulated fund of this year is equal to the accumulated fund of previous year plus the balance of total income and expenditure [NPS (2018), p43].

$$F_t + I_{F_t} + \sum_{m_t} r_t B_t - \sum_{p_t} P_t - E_t = F_{t+1} \tag{3.4.4}$$

where F_t is accumulated fund, I_{F_t} is investment income of accumulated fund, $\sum_{m_t} r_t B_t$ is total contributory income, $\sum_{p_t} P_t$ is total pension expenditure, E_t is total other expenses, and *t* is time.

Accumulated fund currently occupies 37.2% of national GDP (Gross Domestic Product) [NPS (2018), p77]. The accumulated investment return was 5.24% from 1988 to 2018 [NPRI (2018), p23]. However, the trend of return was quite volatile. As for total contributory income and total pension expenditure, the procedure of estimation is as follows. Starting point is the projection of population. By applying labor participation rate and participation rate, the number of national pension members is estimated. It is worth mentioning that the national pension members are categorized as individually insured participant and workplace-based participant. The ratio of income of individually insured to that of workplace-based participants is currently around 50%. Among members, pensioner is basically required to have more than 10 year's participation period and reach age 60 at the beginning (it will be changed to age 65 in 2033). In addition, the number of people who get other benefits such as survivor's pension, disability pension, and lump sum refund is estimated by relevant risk rates. Overlapping cases among variables are adjusted accordingly. Then, calculation for total income and expenditure is implemented with several assumptions such as wage increase rate, contribution exemption rate, collection rate, the proportion of individually insured participants and ratio of income of individually insured to that of workplace-based participants. Economic assumptions like interest rate, inflation rate affect future average income and basic pension amount. The ratio of burden for government on administrative fee, credit for maternity leave and military service and benefit connected to special occupation retirement pension are additional assumptions. Complexity of rates is various from unit rate to age and gender specific rate. Assumption is based on time series analysis and expert's judgment. As vast range of assumptions is applied, the uncertainty of results is inevitably enormous. It is required to carry out sensitivity test of parameters for comprehensive understanding.

Table 20: Illustration on actuarial valuation about peak and depletion of fund, required contribution rate for target fund-to-expenditure ratio of 2, and Pay-As-You-Go (PAYG) rate

	Deficit	***Depletion***	***Ratio of 2***	***PAYG rate***
4th Actuarial Projection(2018)	2042	2057	16.3% (2088)	28.8% (2088)
4th Actuarial Projection(2013)	2044	2059	14.7% (2088)	26.7% (2088)
3rd Actuarial Projection(2013)	2044	2060	12.9% (2083)	22.9% (2083)
3rd Actuarial Projection(2008)	2042	2058	14.1% (2083)	23.5% (2083)
2nd Actuarial Projection(2008)	2043	2060	12.2% (2070)	23.2% (2070)
1st Actuarial Projection(2003)	2044	2060	12.4% (2070)	26.3% (2070)
1st Actuarial Projection(2003)	2036	2047	19.9% (2070)	39.1% (2070)

Source: National Pension Service [NPS (2003), p71| (2008), p151| (2013), p148| (2018), p142]

In Table 20, parenthesis on the first column addresses the parameters of the year. Therefore it is capable to distinguish whether the difference of result is due to parametric change or structural change of pension. Referring to the last two rows, the drastic change reflects the change of the coefficient of benefit c_t from 1.8 to 1.2 (1/3 decrease). Both actuarial valuations belong to 1st actuarial projection. The deficit column shows the first year when loss in the balance occurs, in other words, right after the peak of fund is reached. The depletion column indicates the first year when exhaustion of fund happens. The meaning of target fund-to-expenditure ratio of 2 (Ratio of 2) is that target fund is secured in the last year of projection period to pay expenditure of two years. Rate of Pay-As-You-Go (PAYG) system cost is total pension expenditure divided by total contributory income. Parenthesis on the last two columns addresses the last year of projection period. Actuarial projection covers long span over generation, ordinary 70 years.

Even though the projection year of deficit and depletion of fund seems to be stable, the specific amount of fund at these points shows wide variations in actuarial projection reports. It inevitably implies uncertainty because it is based on various demographic, socioeconomics assumptions at the time of valuation.

The essential issue is that the interval between deficit and depletion of fund is less than 15 years. Comparing with 54 years accumulation phase from 1988 to 2042, it is a sharp contrast. When the accumulated fund has to be liquidated very quickly, the massive aftermath on economy is enormous and the value of asset plummets. Therefore, the focal point should be on how to smoothen the downward trend of fund.

Table 21: Illustration on required contribution rate depends on the target fund-to-expenditure ratio and timing of rise (The last year of projection period is 2088)

	Ratio of 1	***Ratio of 2***	***Ratio of 5***	***No Deficit***	***Constant Ratio***
2020	16.0%	16.3%	17.0%	18.2%	20.2% (17.3)
2030	18.0%	18.3%	19.3%	20.2%	22.2% (14.0)
2040	20.9%	21.4%	22.7%	23.0%	24.9% (10.0)

Source: National Pension Service [NPS (2018), p100]

The meaning of target fund-to-expenditure ratio of 1 (Ratio of 1) is that target fund is secured in the last year of projection period 2088 to pay expenditure of one year. Likewise, the meaning of target fund-to-expenditure ratio of 5 (Ratio of 5) is that target fund is secured in the last year of projection period 2088 to pay expenditure of five years. No deficit means that the accumulated fund will increase until the last year of projection period 2088. Under constant ratio, the long-term fund-to-expenditure ratio is at steady-state condition in the last year of projection period 2088. It can read from the data presented in Table 21 that when timing of rise on contribution rate is prolonged, the required contribution rate is higher, meaning that future generation bear heavier burden. For example, the required contribution rate is 17.0% in 2020 but 19.3% in 2030 in order to achieve the target fund-to-expenditure ratio of 5 in 2088. Moreover, the required contribution rate is 22.7% in 2040 in order to achieve the target fund-to-expenditure ratio of 5 in 2088, which is more demanding as the difference 3.4% (22.7%-19.3%) is higher than 2.3% (19.3%-17.0%). Note that parenthesis on the last column addresses the target fund-to-expenditure ratio in 2088. In practice the pension institution seems to consider Gross Domestic Product (GDP) as a main barometer, therefore relevant information needs to be provided from actuarial projection report. The ratio of accumulated fund to GDP will reach a peak 48.2% in 2034 and decrease with negative acceleration. Applying the scenario, the peak of the ratio is 98.9%, 101.9%, 112.0%, 128.8%, and 170.0% for each. The ratio of total expenditure to GDP equals to 1.3% in 2018 and will reach around 9% in 2070. In addition, the rate of Pay-As-You-Go (PAYG) system cost is 5.2% in 2020 and will reach 29% in long run. The ratio of total contributory income to GDP keep stable around 28% to 32%.

A surge of contribution rate does not guarantee public acceptability. Gradual increase every 5 years depending on the interim results is desirable. In fact, the ratio of 1 and 2 is considered not to be effective target because the depletion happens almost certainly before next actuarial valuation in 5 years. At least the ratio of 5 should get a foothold in the criteria of sustainability of pension. It secures enough time to take prompt action toward public pension while minimizing negative impact.

The elderly dependency ratio leaps from 38.2% to 77.6% between 2030 and 2050. Since the demographic change in the near future is almost unavoidable, there is a strong reason to raise contribution rate before 2030 to mitigate the unprecedented impact on public pension fund. The contribution rate is ultimately determined by the trend of two factors. Therefore, contribution rate should be pegged to the speed of trend on population and dependency ratio in the long run; in this case it can play a role of automatic stabilizer of pension.

It is worth mentioning that the interest rate has a considerable importance on projection. Considering mortality is already quite low and reaches saturation point, high interest rate carried by growing economy will lessen the burden of liability on public pension. Furthermore, investment performance will have more impact on projection when accumulated fund is expected to constitute greater portion of total income in public pension within decades. In case of fund deficit, it is important to set up the standard how liquidation process of fund should be implemented.

4. Conclusion

During the first quarter of 21st century, how to live and design a life after retirement becomes in the spotlight around the world as life expectancy at age 65 is long enough and keep increasing. In particular, the demographic situation of Republic of KOREA is expected to be transformed from the youngest to oldest country in the world just within one century. Longevity risk along with life expectancy is imperative to be dealt with by both individuals and governments.

The thesis provides models and analysis that help to find optimal solution for individual and best solution for policymaker on real-world decision-making process. As shown in Chapter 2, the various illustrations prepare for making a wise decision on individual level. As shown in Chapter 3, public pension plays a vital role to establish welfare country and policymaker equipped with precise analysis can make informed decision on national level. Both parts are important because one part cannot take full responsibility of retirement in modern society.

Two historical cases are first arranged to draw attention. Leonardo Fibonacci equation shows not only how much required retirement fund has been significantly increased over 20 years due to low real interest rate and high life expectancy, but also how much retirement fund is necessary for single man, single woman and married couple. Edmond Henry annuity equation provides criteria for decision between annuity and lump sum amount. From national perspective, the annuity factor can be employed for estimating overall expenditures in respect to individual's retirement benefit. The exchange ratio, i.e. the ratio of consumption amount to saving amount, is presented along with four scenario of interest rate over 50 years. In addition, there is an attempt to reflect mortality rate in the calculation of exchange ratio. It gives a chance to think about the golden ratio of consumption amount to saving amount at present. Ruin probability model as pure probabilistic approach assists setting up the strategy of the combination of coverage ratio and initial wealth.

Through Constant Force of Mortality (CFM) assumption and OLS (Ordinary Least Squares) method, we keep track of the mortality improvement of cohort. It verifies the prospect that life expectancy at age 65 has been increased as survival rate has been improved. However, the mortality improvement expected to be limited as the mortality itself is already low enough considering technology development. Before assessing public pension, projection on population and dependency ratio are preceded beforehand. Full-scale analysis starts from assessing the number of new-born babies, which depends on Total Fertility Rate (TFR). Combining mortality rate and net international migration rate together, it is possible to produce the projection on population and dependency ratio comparable with previous actuarial valuation reports. Afterwards, the impact on population and peak dependency ratio is measured by linear regression. Next, the public pension benefit formula is thoroughly analyzed and the prospects on the indexes of public pension fund and required contribution rate by scenarios are discussed in detail.

Model is a simplified reality. It is not clear how to decompose complicated reality into several key factors with appropriate methodology to make a model for projection. For choosing model, opportunity cost is an important criterion. Therefore, designing model is the art of actuarial judgment. Analyzing the result from the model is also very important. Models can only serve as a support and should not be expected to replace sound personal judgement and experience [ILO (1997), p79].

Restructuring population shape by uplifting birth rate is the centerpiece of sustainable of pension system in the long run with the help of economic growth and investment performance of pension fund. Moreover, the demographic cliff is inevitable destiny for Republic of KOREA, because of that there is a need for a solution on how to deal with the massive financial shock on the public pension within decades. Smooth transition from partial-fund system to Pay-As-You-Go (PAYG) system will lead the success of public pension.

Actuaries play a key role in the design, implementation and operation of social security schemes. Their expertise is an important contribution to the decision-making process in this respect. Providing clear and accessible information also improves public confidence in a social security scheme and is likely to reinforce public and political support [ISSA (2016), p54].

Actuarial valuation indicates that current public pension is difficult to secure continued solvency. Even though the fact is politically sensitive, it should be gravely taken the prospects that the national pension fund will be depleted by 2060. Republic of KOREA confronts watershed period in terms of public pension system.

Appendix

A1. Leonardo Fibonacci Equation Derivation

$$W_0(1+r)-c=W_1 \qquad W_{t-1}=c(1+r)^{-1}, \qquad \text{(A1.1)}$$
$$W_1(1+r)-c=W_2 \qquad W_{t-2}=c(1+r)^{-1}+c(1+r)^{-2}$$
$$\cdots \qquad \cdots$$
$$W_{t-2}(1+r)-c=W_{t-1} \qquad W_{t-n}=\sum_{i=1}^{n}c(1+r)^{-i}$$
$$W_{t-1}(1+r)-c=W_t=0$$

$$W=W_0=\sum_{i=1}^{t}c(1+r)^{-i}=\frac{c(1+r)^{-1}((1+r)^{-t}-1)}{(1+r)^{-1}-1}=\frac{c((1+r)^{-t}-1)}{1-(1+r)}=\frac{c}{r}(1-(1+r)^{-t})$$
$$\xrightarrow{\textit{Linear Approximation}}\frac{c}{r}(1-(e^r)^{-t})=\frac{c}{r}(1-e^{-rt})=\frac{1}{r}(c-\frac{c}{e^{tr}}) \qquad \text{(A1.2)}$$
$$W=\frac{c}{r}(1-e^{-rt})\rightarrow\frac{Wr}{c}=(1-e^{-rt})\rightarrow e^{-rt}=\frac{c-Wr}{c}$$
$$\therefore t=\frac{1}{r}\ln\left(\frac{c}{c-Wr}\right) \qquad \blacksquare$$

$$\text{Linear Approximation: } f(x)\approx f(a)+f'(a)(x-a)$$
$$f(r)=\ln(1+r), f'(r)=\frac{1}{1+r}, \qquad f(0)=0, f'(0)=1$$
$$\therefore \ln(1+r)\approx r, \qquad 1+r\approx e^r$$

On the left side of formula (A1.1), the relationship about the wealth of sequential time is illustrated. Supposing that the wealth is depleted after time t, the general equation of wealth along with time such as W_{t-1}, W_{t-2} are inferred in reverse order on the right side of formula (A1.1). After t times of reverse steps from W_t, initial wealth W_0 equal to required retirement funds W is derived as follow: $W_{t-t}=W_0=\sum_{i=1}^{t}c(1+r)^{-i}$. It is geometric series in discrete time basis. In order to present the formula in continuous time basis, linear approximation method in box above is applied. The more tangent point a is closer to 0, the function *ln(1+r)* is more closer to the tangent line r. Setting tangent point a is 0, e^r is a good substitute for *ln(1+r)* and the formula of initial wealth W_0 is derived. Lastly, the formula of t is obtained by rearrangement of the formula (A1.2).

A2. *Leonardo Fibonacci Equation Technical Analysis*

$$W = \frac{1}{r}\left(c - \frac{c}{e^{tr}}\right) = \frac{c(e^{tr}-1)}{re^{tr}}, \qquad \text{Quotient Rule:} \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}$$

$$W'(r) = \frac{dW}{dr} = \frac{cte^{tr}(re^{tr}) - c(e^{tr}-1)(e^{tr}+tre^{tr})}{(re^{tr})^2} = \frac{ctre^{2tr} - c(e^{2tr} + tre^{2tr} - e^{tr} - tre^{tr})}{(re^{tr})^2}$$

$$= \frac{ce^{tr} + ctre^{tr} - ce^{2tr}}{(re^{tr})^2} = \frac{c + ctr - ce^{tr}}{r^2e^{tr}} = c\left(\frac{1 + tr - e^{tr}}{r^2e^{tr}}\right) < 0 \qquad \text{(A2.1)}$$

$$f(x) = 1 + x - e^x, \qquad f(0) = 0$$

$$f'(x) = 1 - e^x < 0, \qquad \forall x > 0, \qquad f'(0) = 0$$

$$\therefore f(x) = 1 + x - e^x < 0, \qquad \forall x > 0 \qquad f(tr) = 1 + tr - e^{tr} < 0, \qquad \forall\, t, r > 0$$

$$W'(t) = \frac{dW}{dt} = \frac{cre^{tr}(re^{tr}) - c(e^{tr}-1)(r^2e^{tr})}{(re^{tr})^2} = \frac{cre^{tr} - c(e^{tr}-1)r}{re^{tr}} = \frac{cr}{re^{tr}} = \frac{c}{e^{tr}} > 0 \qquad \text{(A2.2)}$$

The relationship between two variables is clarified by partial differentiation. Quotient rule is applied. By analyzing the numerator of formula (A2.1) in box above, it is clear to see that the derivative of wealth *W* by interest rate *r* is negative. Likewise, the formula (A2.2) shows that derivative of wealth *W* by life expectancy *t* is positive. Therefore, the required wealth *W* will be increased when interest rate *r* decreases and life expectancy *t* increases.

B. *Macaulay Duration of Life Annuity and Annuity Certain*

$$\text{Macaulay Duration: } \frac{\sum_{t=1}^{\infty} t * PV_t}{\sum_{t=1}^{\infty} PV_t}, \qquad v = \frac{1}{1+r} < 1$$

$$\text{Life Annuity: } a_x = \sum_{t=1}^{w-x} v^t p_{x,t} \qquad \text{Annuity Certain: } a_{\overline{w-x|}} = \sum_{t=0}^{w-x} v^t$$

$$if\ w = \infty, p_{x+t} = p\ \ \forall t, \qquad a_x = \sum_{t=1}^{\infty} v^t p^t \qquad if\ w = \infty, \qquad a_{\overline{\infty|}} = \sum_{t=0}^{\infty} v^t$$

$$\text{Macaulay Duration of Life Annuity: } \frac{vp + 2v^2p^2 + \cdots}{vp + v^2p^2 + \cdots} = \frac{\frac{vp}{(1-vp)^2}}{\frac{vp}{1-vp}} = \frac{1}{1-vp}, \qquad p < 1 \tag{B.1}$$

$$\text{Macaulay Duration of Annuity Certain: } \frac{v + 2v^2 + \cdots}{v + v^2 + \cdots} = \frac{\frac{v}{(1-v)^2}}{\frac{v}{1-v}} = \frac{1}{1-v} > \frac{1}{1-vp} \tag{B.2}$$

For simplicity of calculation, time *t* is infinite and survival rate *p* is constant. Macaulay Duration of life annuity is smaller than of annuity certain since survival rate *p* is less than 1.

C. *The Proof of the Ruin Probability Formula*

$$k=0,\qquad \sum_{t=1}^{\infty}(1-p)p^{t-1}(1-q)^t=(1-p)\{(1-q)+p(1-q)^2+\cdots\}$$

$$=(1-p)\left[\frac{1-q}{1-\{p(1-q)\}}\right]=\frac{(1-p)(1-q)}{1-p+pq}=\left(1-\frac{q}{1-p+pq}\right)^1$$

$$\therefore \sum_{t=1}^{\infty}(1-p)\binom{t-1}{0}p^{t-1-0}(1-p)^0(1-q)^t=\left(1-\frac{q}{1-p+pq}\right)^{0+1} \tag{C.1}$$

$$k=1,\sum_{t=2}^{\infty}(1-p)(t-1)p^{t-2}(1-p)(1-q)^t=(1-p)^2(1-q)^2\{1+2p(1-q)+\cdots\}$$

$$=(1-p)^2(1-q)^2\left[\frac{1}{1-\{p(1-q)\}}+\frac{p(1-q)}{1-\{p(1-q)\}}+\cdots\right]=\frac{(1-p)^2(1-q)^2}{[1-\{p(1-q)\}]^2}$$

$$\therefore \sum_{t=2}^{\infty}(1-p)\binom{t-1}{1}p^{t-1-1}(1-p)^1(1-q)^t=\left(1-\frac{q}{1-p+pq}\right)^{1+1} \tag{C.2}$$

$$k=2,\sum_{t=3}^{\infty}(1-p)\frac{(t-1)(t-2)}{2}p^{t-3}(1-p)^2(1-q)^t=(1-p)^3(1-q)^3\{1+3p(1-q)+\cdots\}$$

$$\xRightarrow{m=t-2}\frac{(1-p)^3(1-q)^3}{2}\sum_{m=1}^{\infty}(m^2+m)\,p^{m-1}(1-q)^{m-1}$$

$$=\frac{(1-p)^3(1-q)^3}{2}\left\{\sum_{m=1}^{\infty}m^2\,p^{m-1}(1-q)^{m-1}+\sum_{m=1}^{\infty}m\,p^{m-1}(1-q)^{m-1}\right\}$$

$$=\frac{(1-p)^3(1-q)^3}{2}\left[\frac{1+p(1-q)}{[1-\{p(1-q)\}]^3}+\frac{1}{[1-\{p(1-q)\}]^2}\right]=\frac{(1-p)^3(1-q)^3}{[1-\{p(1-q)\}]^3}$$

$$\therefore \sum_{t=3}^{\infty}(1-p)\binom{t-1}{2}p^{t-1-2}(1-p)^2(1-q)^t=\left(1-\frac{q}{1-p+pq}\right)^{2+1} \tag{C.3}$$

$$\therefore \sum_{t=1}^{\infty}(1-p)\binom{t-1}{k}p^{t-1-k}(1-p)^k(1-q)^t=\left(1-\frac{q}{1-p+pq}\right)^{k+1}\quad k=0,1,2,\cdots \quad ■$$

To prove the ruin probability formula, mathematical induction method is adopted for integer *k* equal or greater than 0. In (C.1), infinite geometric series is visible. Applying to the sum of infinite geometric series formula $\frac{a}{1-r}$, *a* is *1-q* and *r* is *p(1-q)*. In (C.2), infinite geometric series of sum of infinite geometric series appears, *a* is $\frac{1}{1-\{p(1-q)\}}$ and *r* is *p(1-q)*. In (C.3), power series emerges. In this way, the ruin probability formula is inferred for integer *k* greater than 2.

D. R Estimation about Mortality Improvement

```
Man Residuals:
    Min         1Q        Median      3Q        Max
-0.031978 -0.012018 -0.001048  0.008413  0.063861
Coefficients:
                Estimate    Std. Error  t value   Pr(>|t|)
(Intercept) -0.068025   0.006155  -11.05 2.79e-14 ***
Time          0.013064   0.000228   57.29  < 2e-16 ***
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.02053 on 44 degrees of freedom
Multiple R-squared: 0.9868,    Adjusted R-squared: 0.9865
F-statistic: 3282 on 1 and 44 DF, p-value: < 2.2e-16
```

```
Woman Residuals:
    Min         1Q        Median      3Q        Max
-0.032673 -0.021734 -0.006668  0.023556  0.047779
Coefficients:
                Estimate     Std. Error   t value    Pr(>|t|)
(Intercept) -0.0504545  0.0074240  -6.796 2.28e-08 ***
Time          0.0089759  0.0002751  32.633  < 2e-16 ***
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.02477 on 44 degrees of freedom
Multiple R-squared: 0.9603, Adjusted R-squared:  0.9594
F-statistic: 1065 on 1 and 44 DF, p-value: < 2.2e-16
```

E. *Calculation of Appropriate Contribution Rate*

$$r(12B)\ddot{a}_{65-d:\overline{d}|} = v^d p_{65-d,d}\ddot{a}_{65}\frac{12}{d}\big(1+0.05(d-20)\big)\sum_t c_t(a_tA+b_tB)I_t$$

$$+r(12B)\left(\ddot{a}_{\overline{1}|}vq_{65-d}+\ddot{a}_{\overline{2}|}v^2p_{65-d}q_{65-(d-1)}+\cdots+\ddot{a}_{\overline{d}|}v^dp_{65-d,d-1}q_{65-\{d-(d-1)\}}\right)$$

$$P=\frac{1}{d}\big(1+0.05(d-20)\big)\sum_t c_t(a_tA+b_tB)I_t$$

$$rB\big(1+vp_{65-d}+\cdots+v^{d-1}p_{65-d,d-1}\big)=v^dp_{65-d,d}\ddot{a}_{65}P$$

$$+rB\big[(1+i)vq_{65-d}+\{(1+i)^2+(1+i)\}v^2p_{65-d}q_{65-(d-1)}$$

$$+\cdots\{(1+i)^d+\cdots+(1+i)\}v^dp_{65-d,d-1}q_{65-\{d-(d-1)\}}\big]\cdots(*) \tag{E.1}$$

$$(*)= rB\{\big(1-p_{65-d,d}\big)+v\big(p_{65-d}-p_{65-d,d}\big)+\cdots+v^{d-1}\big(p_{65-d,d-1}-p_{65-d,d}\big)\} \tag{E.2}$$

$$rBp_{65-d,d}\big(1+v+\cdots+v^{d-1}\big)=v^dp_{65-d,d}\ddot{a}_{65}P, \qquad \ddot{a}_{\overline{d}|}=1+v+\cdots+v^{d-1}$$

$$\therefore r=\frac{v^d\ddot{a}_{65}P}{B\ddot{a}_{\overline{d}|}} \qquad \blacksquare$$

When APV (Actuarial Present Value) of premium and benefit are equal, actuarial equivalence principle is fulfilled. At the point, contribution rate *r* is actuarially appropriate. Expanding both side of the equation is a necessary step for simplification of calculation. The (*) part in equation (E.1) simplified by (E.2) is reduced to *d* years certain annuity by the left side of equation (E.1). By dividing the coefficient of *r*, appropriate contribution rate is drawn.

F. Polynomial Power Series Triangle (Eulerian number)

The invention of polynomial power series triangle stems from the process of proving the formula (2.5.1) in Appendix C. From the fact that the left of the formula (2.5.1) consists of k dimensions polynomial power series and generalized equation on the right of the formula (2.5.1) for integer k, it is possible to derive the coefficients of k dimensions power series manually as below. Indeed, the generalized coefficients a_{Ii}, b_{Jj} are inferred from these coefficient numbers.

$$k = 1, q = 0 \sum_{t=2}^{\infty}(1-p)\binom{t-1}{1}p^{t-1-1}(1-p)^1 = \frac{(1-p)^2}{(1-p)^2}$$

$$\sum_{t=2}^{\infty}(1-p)(t-1)p^{t-2}(1-p) \xRightarrow{m=t-1} (1-p)^2 \sum_{m=1}^{\infty} mp^{m-1} = \frac{(1-p)^2}{(1-p)^2}$$

$$\therefore \sum_{m=1}^{\infty} mp^{m-1} = \frac{1}{(1-p)^2} \tag{F.1}$$

$$k = 2, q = 0 \sum_{t=3}^{\infty}(1-p)\binom{t-1}{2}p^{t-1-2}(1-p)^2 = \frac{(1-p)^3}{(1-p)^3}$$

$$\sum_{t=3}^{\infty}(1-p)\frac{(t-1)(t-2)}{2!}p^{t-3}(1-p)^2 \xRightarrow{m=t-2} (1-p)^3 \sum_{m=1}^{\infty} \frac{(m+1)m}{2} p^{m-1} = \frac{(1-p)^3}{(1-p)^3}$$

$$\therefore \sum_{m=1}^{\infty} m^2p^{m-1} = \frac{2}{(1-p)^3} - \sum_{m=1}^{\infty} mp^{m-1} = \frac{1+p}{(1-p)^3} \tag{F.2}$$

$$k = 3, q = 0 \sum_{t=4}^{\infty}(1-p)\binom{t-1}{3}p^{t-1-3}(1-p)^3 = \frac{(1-p)^4}{(1-p)^4}$$

$$\sum_{t=4}^{\infty}(1-p)\frac{(t-1)(t-2)(t-3)}{3!}p^{t-4}(1-p)^3 \xRightarrow{m=t-3} (1-p)^4 \sum_{m=1}^{\infty} \frac{(m+2)(m+1)m}{6} p^{m-1} = \frac{(1-p)^4}{(1-p)^4}$$

$$\therefore \sum_{m=1}^{\infty} m^3p^{m-1} = \frac{6}{(1-p)^4} - 3\sum_{m=1}^{\infty} m^2p^{m-1} - 2\sum_{m=1}^{\infty} mp^{m-1} = \frac{1+4p+p^2}{(1-p)^4} \tag{F.3}$$

$$\sum_{m=1}^{\infty} m^4\, p^{m-1} = \frac{1+11p+11p^2+p^3}{(1-p)^5} \tag{F.4}$$

$$\sum_{m=1}^{\infty} m^5\, p^{m-1} = \frac{1+26p+66p^2+26p^3+p^4}{(1-p)^6} \tag{F.5}$$

$$\sum_{m=1}^{\infty} m^6\, p^{m-1} = \frac{1+57p+302p^2+302p^3+57p^4+p^5}{(1-p)^7} \tag{F.6}$$

$$\sum_{m=1}^{\infty} m^7\, p^{m-1} = \frac{1+120p+1191p^2+2416p^3+1191p^4+120p^5+p^6}{(1-p)^8} \tag{F.7}$$

$$\sum_{m=1}^{\infty} m^8\, p^{m-1} = \frac{1+247p+4293p^2+15619p^3+15619p^4+4293p^5+247p^6+p^7}{(1-p)^9} \tag{F.8}$$

$$\cdots$$

$$\begin{array}{c}
a_{11} \\
a_{21}a_{22} \\
a_{31}a_{32}a_{33} \\
a_{41}a_{42}a_{43}a_{44} \\
a_{51}a_{52}a_{53}a_{54}a_{55} \\
a_{61}a_{62}a_{63}a_{64}a_{65}a_{66} \\
a_{71}a_{72}a_{73}a_{74}a_{75}a_{76}a_{77} \\
a_{81}a_{82}a_{83}a_{84}a_{85}a_{86}a_{87}a_{88} \\
a_{91}a_{92}a_{93}a_{94}a_{95}a_{96}a_{97}a_{98}a_{99} \\
\cdots
\end{array}
\qquad
\begin{array}{c}
b_{21}b_{22} \\
b_{31}b_{32}b_{33}b_{34} \\
b_{41}b_{42}b_{43}b_{44}b_{45}b_{46} \\
b_{51}b_{52}b_{53}b_{54}b_{55}b_{56}b_{57}b_{58} \\
b_{61}b_{62}b_{63}b_{64}b_{65}b_{66}b_{67}b_{68}b_{69}b_{610} \\
b_{71}b_{72}b_{73}b_{74}b_{75}b_{76}b_{77}b_{78}b_{79}b_{710}b_{711}b_{712} \\
b_{81}b_{82}b_{83}b_{84}b_{85}b_{86}b_{87}b_{88}b_{89}b_{810}b_{811}b_{812}b_{813}b_{814} \\
\cdots
\end{array}$$

$$I = 1,2,\cdots, i = 1,2,\cdots, I \qquad J = 2,3,\cdots, j = 1,2,\cdots,2(J-1) \tag{F.9}$$

$$a_{Ii} = \begin{cases} a_{II+1-i}, symmetry \\ 1, when\ i\ is\ 1 \\ a_{I-1i-1}b_{I-12i-3} + a_{I-1i}b_{I-12i-2} \end{cases} \qquad b_{Jj} = \begin{cases} b_{J2J-1-j}, symmetry \\ J - \left(\dfrac{j-1}{2}\right), when\ j\ is\ odd \\ 2 + \left(\dfrac{j-2}{2}\right), when\ j\ is\ even \end{cases}$$

a_{Ii}

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

1 120 1191 2416 1191 120 1

1 247 4293 15619 15619 4293 247 1

1 502 14608 88234 156190 88234 14608 502 1

…

b_{Jj}

2 2

3 2 2 3

4 2 3 3 2 4

5 2 4 3 3 4 2 5

6 2 5 3 4 4 3 5 2 6

7 2 6 3 5 4 4 5 3 6 2 7

8 2 7 3 6 4 5 5 4 6 3 7 2 8

…

These coefficients a_{Ii} illustrate the numerator of k dimension power series. As the cofactors b_{Jj} links two consecutive layer of a_{Ii}, $k+1$ dimension can be developed. Hence, these formulas (F.9) embody the general pattern of the coefficients and cofactors for power series.

I elaborated on tracking the coefficients of numerator for k dimensions power series. There is remarkable relationship between consecutive layer of coefficients as presented above; both layers are connected by cofactors. Both coefficients a_{Ii} and cofactors b_{Jj} shape triangle and each layer of cofactors is relevant to calculate next layer of coefficients. The name "Polynomial Power Series Triangle" is used in this thesis since the shape follows "Pascal's Triangle". In literature, these coefficients a_{Ii} have been given a name of "Eulerian number". Comtet defines Eulerian numbers as the number of permutation runs of length k-1 and derives "Eulerian number" [Comtet (1974), p243].

By these formulas (F.9), it is much easier to find the coefficient of high degree of polynomial power series. Arithmetic with the complicated polynomial power series becomes significantly simple and understandable.

F1. Polynomial Power Series Triangle Application-Geometric Distribution

The number of trial is *x* with the number of failure is *x-1* and the number of success is *1*. The probability of success is p.

$$X \sim Geometric(p), \qquad Pr(X=x) = p(1-p)^{x-1}, \qquad x = 0,1,2,\cdots$$

$$E[X] = \frac{1}{p}, \qquad Var[X] = \frac{1-p}{p^2}$$

$$M_x[t] = E(e^{tx}) = \frac{pe^t}{1-qe^t}, \qquad q = 1-p$$

$$E[X] = 1p + 2p(1-p) + 3p(1-p)^2 + \cdots$$

$$= p(1 + 2(1-p) + 3(1-p)^2 + \cdots) = p\left(\frac{1}{p^2}\right) = \frac{1}{p} \tag{F1.1}$$

$$E[X^2] = 1^2p + 2^2p(1-p) + 3^2p(1-p)^2 + \cdots$$

$$= p(1^2 + 2^2(1-p) + 3^2(1-p)^2 + \cdots) = p\left(\frac{2-p}{p^3}\right) = \frac{2-p}{p^2} \tag{F1.2}$$

$$E[X^3] = 1^3p + 2^3p(1-p) + 3^3p(1-p)^2 + \cdots = p(1^3 + 2^3(1-p) + 3^3(1-p)^2 + \cdots)$$

$$= p\left(\frac{1 + 4(1-p) + (1-p)^2}{p^4}\right) = \frac{1 + 4(1-p) + (1-p)^2}{p^3} \tag{F1.3}$$

$$E[X^4] = \frac{1 + 11(1-p) + 11(1-p)^2 + (1-p)^3}{p^4} \tag{F1.4}$$

$$E[X^5] = \frac{1 + 26(1-p) + 66(1-p)^2 + 26(1-p)^3 + (1-p)^4}{p^5} \tag{F1.5}$$

Geometric distribution is the simplest power series. With the help of “Polynomial Power Series Triangle” (Eulerian number), the formula of high degree power moments is easily derived. Even though formulas are illustrated until 5th power moment here from (F1.1) to (F1.5), it is clear to determine all formula of *k* power moment higher than 5.

Moment generating function is basic ingredient to calculate the formula of *k* power moment. However, it requires differentiating moment generating function *k* times to get the formula of *k* power moment. For example, it is cumbersome task to differentiate moment generating function of geometric distribution $\frac{pe^t}{1-qe^t}$ for more than 10 times. In this case, using “Polynomial Power Series Triangle” (Eulerian number) is outstandingly faster and more convenient than moment generating function. Ultimately, any *k* degree complicated polynomial equation of variable *x* can be disassembled into *k+1* power moments to calculate expected value.

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